ACCURACY OF NUMERICAL POLICIES FOR DYNAMIC MODELS WITH OCCASIONALLY BINDING CONSTRAINTS

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ABSTRACT. This paper uses the Lebesgue Theorem to derive a computable policy error bound from given value function error bounds for numerical solutions to a widely used savings problem that has an occasionally binding borrowing constraint. The bound can be used to evaluate the accuracy of numerical policies and applies even when the optimal policy function is non-interior.

Keywords: Error bounds, constrained optimization, Lebesgue Theorem, savings problem

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1. INTRODUCTION

This paper is concerned with the accuracy of numerically obtained policy functions for dynamic programming problems with occasionally binding constraints. In particular, we derive policy error bounds for numerical solutions to a savings problem, in which an inequality constraint arises from a borrowing limit. Studied by Schechtman and Escudero (1977), this problem and its various versions appears in macroeconomics textbooks such as Ljungqvist and Sargent (2004) and is embedded in many quantitative macroeconomic workhorse models, e.g. Deaton (1991), Huggett (1993), Aiyagari (1994) and Krusell and Smith (1998).

Our focus on policy error bounds is motivated by its relevance for quantitative analysis. Approximate policy functions are used in equilibrium computation, counterfactual exercises and estimation. A knowledge of the approximation error is thus useful for drawing reliable quantitative conclusions.

Within the macroeconomics literature, it has become a common practice to use the Euler equation residual as a measure of accuracy for numerical solutions (See comments in Aruoba, Fernandez-Villaverde, and Rubio-Ramirez (2006)). Santos (2000) provides a theoretical foundation for this approach by showing that for the stochastic growth model, the value and policy error bounds are of the same order of magnitude as the residuals. However, this result relies on the interiority of the optimal policy function, which does not hold in any problem with nontrivial inequality constraints. On the other hand, computable value error bounds exist for popular algorithms. For example, for a general class of dynamic programming problems, Stachurski (2008) derives theoretical value function error bound for the fitted value function algorithm.

Existing research such as Christiano and Fisher (2000) and Pierri (2011) have focused on deriving value error bounds for models with occasionally binding inequality constraints. However, small value errors do not in general imply small policy errors. The current paper contributes to the literature by showing how to derive computable policy error bounds from *any* given value error bound. A key property of the savings problem used in our derivation is the strong concavity of the utility function, which is a standard assumptions in macroeconomics. For example, using the value error bound from Stachurski (2008), our result yields computable policy error bounds for the commonly used fitted value iteration algorithm with linear interpolation.

We also conduct numerical simulations to investigate properties of the error bound. First, as in Santos (2000), the discount factor and the curvature of the utility function have large impact on the tightness of the error bound. Furthermore, we show that for calibration values commonly used in the macroeconomic literature and practical computation time, we can derive very tight error bounds of the policy function evaluated at the neighborhood around the state where the borrowing constraint binds. Hence our error bound may be useful for applications that concerns with identifying the borrowing constrained agents (e.g. Landvoigt, Piazzesi, and Schneider (2010)).

The paper is organized in the following way. Section 2 gives a brief description of the savings problem. Section 3 derives the policy function error bounds and section 4 shows how to compute the value function and policy function error bounds for the fitted value iteration algorithm. Section 5 concludes.

2. A Savings Problem

Consider the savings decision problem of a household facing uncertain fluctuations in its income. The household is risk averse and wants to smooth consumption across periods and states. It can borrow and lend at a constant risk-free rate but borrowing cannot exceed a certain level. Here, we consider a version of the problem due to Huggett (1993).

Given initial asset level and income level (a_0, s_0), the household solves

$$\max_{\{a_{t+1}\}_{t\geq 0}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(a_t + s_t - q a_{t+1}), \quad \text{s.t.} \quad -\phi \leq a_{t+1} \leq \frac{a_t + s_t}{q},$$

where $\beta \in (0,1)$ is the household's discount factor, $q > \beta$ is the price of a unit of credit balance and ϕ is the borrowing limit. The utility function has the CRRA form: $u(c) = \frac{c^{1-\mu}}{1-\mu}$ with $\mu > 1$ or $u(c) = \log(c)$. The income process s_t follows a two-state Markov Chain with transition probability $\pi(s'|s) := Prob(s_{t+1} = s'|s_t = s) > 0$ for $s, s' \in S := \{s_l, s_h\}, s_l < s_h$ and $\pi(s_h|s_h) \geq \pi(s_h|s_l)$. To ensure that the feasible set is nonempty for any realization of income, it is assumed that $-\phi < -\phi + s_l + q\phi$.

The above problem can be solved by dynamic programming. The state space is $X := A \times S$ where $A := [-\phi, \infty)$. For each $x \in X$, the household's feasible set is

$$\Gamma(x) := \{a' \in A : a' \geq -\phi, qa' \leq a + s\}$$

and the set of feasible policies G consists of all bounded Borel measurable functions $g : X \longrightarrow \mathbb{R}$ that satisfy $g(x) \in \Gamma(x)$ for all $x \in X$. Denoting the value of following a policy g by

$$V_g(a_0, s_0) := \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(a_t + s_t - qg(a_t, s_t)),$$

the value function V^* is defined as

$$V^*(a, s) := \sup_{g \in G} V_g(a, s), \quad (a, s) \in X,$$

and the optimal policy function g^* is the feasible policy that satisfies

$$V_{g^*} = V^*.$$

Huggett (1993) shows that there exists a unique $V^*(a, s)$, which is strictly increasing, strictly concave and continuously differentiable in a, and satisfies

$$V^*(a, s) = \max_{a' \in \Gamma(a,s)} u(a + s - qa') + \beta \sum_{s' \in S} V^*(a', s')\pi(s'|s), \quad (a, s) \in X.$$

Furthermore, an optimal policy function g^* exists and satisfies

$$V^*(a, s) = u(a + s - qg^*(a, s)) + \beta \sum_{s' \in S} V^*(g^*(a, s), s')\pi(s'|s), \quad (a, s) \in X.$$

3. Error Bounds

Given a feasible policy function $\hat{g}(a, s)$, we derive an upper bound for $|g^*(a, s) - \hat{g}(a, s)|$ in terms of $|V^*(a, s) - V_{\hat{g}}(a, s)|$. For any (a, s), define

$$F(a') := u(a + s - qa') + \beta \sum_{s' \in S} V^*(a', s')\pi(s'|s), \quad a' \in \Gamma(a, s)$$

and let $a^* := g^*(a, s)$ and $\hat{a} := \hat{g}(a, s)$. Noting that a^* maximizes F on $\Gamma(a, s)$, $\hat{a} \in \Gamma(a, s)$ and $V_{\hat{g}}(a, s) = u(a + s - q\hat{a}) + \beta \sum_{s' \in S} V_{\hat{g}}(\hat{a}, s')\pi(s'|s)$, we can show that $0 \leq F(a^*) - F(\hat{a}) \leq V^*(a, s) - V_{\hat{g}}(a, s)$. In the remainder of this section, we use a generalized Taylor expansion of F around a^* to show that policy function errors are of the same order as $F(a^*) - F(\hat{a})$ and hence $V^*(a, s) - V_{\hat{g}}(a, s)$.

3.1. **When g^* is interior.** To clarify the need to look beyond the Euler equation residual, first let us consider the case when a^* is in the interior of $\Gamma(a,s)$ and V^* is twice differentiable. Here, a second order Taylor expansion tells us that there exists some $c = t\hat{a} + (1-t)a^*$, $t \in (0,1)$ such that

$$F(\hat{a}) - F(a^*) \quad = \quad F'(a^*)(\hat{a} - a^*) + \frac{F''(c)}{2}(\hat{a} - a^*)^2.$$

If a^* is interior, $F'(a^*) = 0$. Furthermore, the concavity V^* implies $|F''(c)| \geq q^2|u''(a+s+q\phi)|$. Therefore, we can bound policy function errors by

$$(\hat{a} - a^*)^2 \leq 2\frac{F(a^*) - F(\hat{a})}{q^2|u''(a+s+q\phi)|}.$$

If $\hat{g}$ is not interior, $F'(a^*) = 0$ may not hold and V^* may not be twice differentiable. As a result of non-interiority, the above derivation does not readily apply.

3.2. **When g^* may be non-interior.** Unfortunately, the interiority and second order differentiability assumptions may not hold in our applications. More precisely, Huggett (1993) shows that there exists $\tilde{a}$ such that $g^*(a, s_l) = -\phi$ for all $a \leq \tilde{a}$ and $g^*(a, s_l) > -\phi$ for all $a > \tilde{a}$. As the value of $\tilde{a}$ is unknown, we cannot derive policy function error bounds by partitioning the state space. Instead, we use a generalized Taylor expansion that only requires V^* to be almost everywhere twice differentiable.

Lemma 3.1. *Given $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\Gamma \subset \mathbb{R}$, assume Γ is closed, convex and nonempty, f is C^1 and strongly concave in the sense that there exists a constant $\eta > 0$ such that $f(\cdot) + h(\cdot)$ is concave for some convex function $h : \mathbb{R} \longrightarrow \mathbb{R}$ with $h''(\cdot) \geq \eta$. Let*

$$z^* = \text{argmax}_{z\in\Gamma} \quad f(z).$$

Then

$$(z - z^*)^2 \leq 2\frac{f(z^*) - f(z)}{\eta}, \quad \forall z \in \Gamma.$$

Proof. The existence and uniqueness of z^* follows from the properties of Γ and the continuity and strict concavity of f. By the definition of integration,

$$\begin{aligned} f(z) - f(z^*) - f'(z^*)(z - z^*) &= \int_{z^*}^{z} f'(t)dt - \int_{z^*}^{z} f'(z^*)dt \\ &= \int_{z^*}^{z} f'(t) - f'(z^*)dt. \end{aligned}$$

As Γ is compact and f is C^1 and strongly concave, f' is Lipschitz continuous on Γ. It is well-known that if a function is Lipschitz continuous, it is absolutely continuous. Then by the Lebesgue theorem [1], f' is almost everywhere differentiable, summable and

$$f'(t) - f'(z^*) = \int_0^1 f''(z^* + r(t - z^*))(t - z^*)dr, \quad t \in [z, z^*]$$

where $f''(y)$ is the usual second order derivative whenever y is a point at which f is twice differentiable. Thus we have a generalized Taylor expansion

(1)

$$f(z) - f(z^*) - f'(z^*)(z - z^*) = \int_{z^*}^{z} \int_0^1 f''(z^* + r(t - z^*))(t - z^*)drdt.$$

Since z^* is the maximizer of f on Γ, it satisfies $f'(z^*)(z - z^*) \leq 0$ for all $z \in \Gamma$. Also, the strong concavity of f further implies that for all $y \in [z, z^*]$ where f is twice differentiable, $f''(y) < -\eta$. Substitution of these into (1) yields

$$f(z^*) - f(z) \geq \int_{z^*}^{z} \eta(t - z^*)dt = \frac{\eta}{2}(z - z^*)^2.$$

□

Noting that u is strongly concave with $u''(a + s - qa') \leq u''(a + s + q\phi)$ for all $a' \in \Gamma(a, s)$, an application of Lemma 3.1 to F yields the following policy error bound:

Theorem 3.1. *For any feasible policy function $\hat{g}$ and $(a, s) \in X$,*

$$(\hat{g}(a,s) - g^*(a,s))^2 \leq 2\frac{V^*(a,s) - V_{\hat{g}}(a,s)}{q^2|u''(a + s + q\phi)|}. \tag{2}$$

Remark 3.1. As the derivation only uses the properties of V^*, our policy error bound holds for value function error bounds derived from any algorithm, i.e., it is algorithm free

[1] Rudin (1987) Theorem 7.20

Remark 3.2. The curvature of the utility function and the discount factor β have large impact on the tightness of the error bound. For example, (2) suggests that the error bound is large at states where the second-order derivative of the utility function is small. Also, the approximate value function is often generated by a contractive operator with modulus β and its approximation error has coefficient $1/(1-\beta)$.

Remark 3.3. Santos (2000) shows that when the solution is interior, value and policy errors associated with policy g are of the same order of magnitude as the Euler equation residual:

$$\epsilon_g := \max_{(a,s)} |qu'(a+s-qg(a,s)) - \beta\mathbb{E}u'(g(a,s)+s'-qg(g(a,s),s'))|.$$

Furthermore, since $\epsilon_g = 0$ is equivalent to g being optimal, $\epsilon_{\hat{g}}$, the residual evaluated at the approximate solution, is informative of the accuracy of $\hat{g}$. However, when the solution is not interior, the residual is positive at the optimal solution and the necessary and sufficient condition for optimality becomes

$$\begin{aligned} h_g \quad := \quad & \max_{(a,s)} |\min\{qu'(a+s-qg(a,s)) - \beta\mathbb{E}u'(g(a,s)+s'-qg(g(a,s),s')), \\ & g(a,s)+\phi\}| = 0. \end{aligned}$$

This suggests that $h_{\hat{g}}$ maybe more informative than $\epsilon_{\hat{g}}$. (work in progress).

4. SIMULATIONS

We show in the Appendix that the savings problem satisfies the assumptions in Stachurski (2008) for deriving *computable* value function error bounds for the fitted value iteration algorithm with piecewise linear interpolation. Figure 1 illustrate the computed policy function and error bound for a calibration of the model in Huggett (1993). The parameter values used are $\beta = 0.99$, $q = 0.997$, $s_h = 1$, $s_l = 0.1$, $\pi_{hh} = 0.925$, $\pi_{lh} = 0.5$, $\phi = 6$, $\sigma = 3$. The grid has 5000 points over $[-6, 5.5]$ and is spaced so that we have a finer grid at the low asset values.

Figure 1 shows that our error bound is very tight the low wage income and low asset value state. In particular, we can very accurately pinpoint where the borrowing constraint binds despite the fact that β is close to 1. For larger asset and wage income values, we do not have meaningful bounds because the utility function is very flat and small

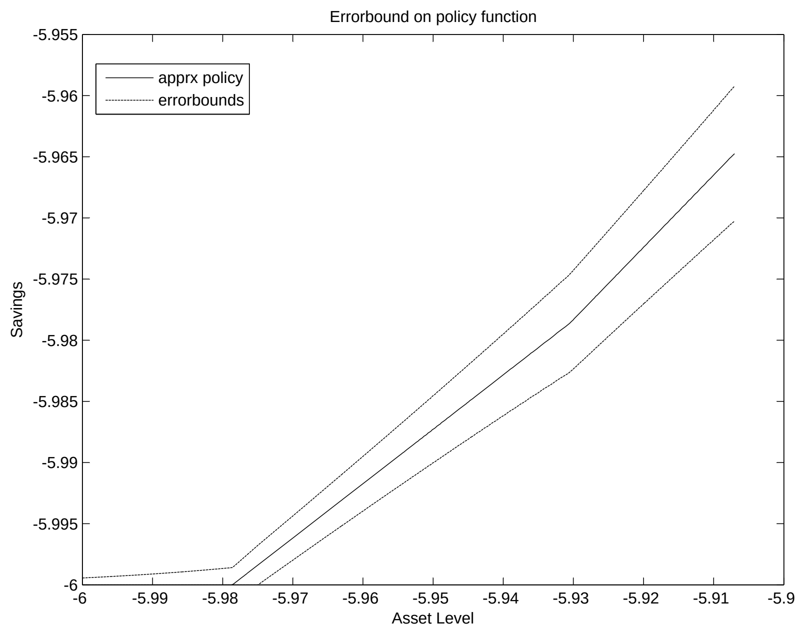

FIGURE 1. Error bounds

value difference can imply large differences in policy. This numerical exercise suggests two potentially useful applications of our error bound. First, there exists economics questions that concerns with the identification of the borrowing constraint agents. For example, Landvoigt, Piazzesi, and Schneider (2010) finds that price volatility of cheaper houses is higher than more expensive houses because the owners of such houses are borrowing constrained. Second, a common method used to save computation time when solving the savings problem is to allocate many grid points around states where the borrowing constraint may bind. One faces a tradeoff between accuracy at the kink and computation time. Our error bound provides a mapping between grid points and accuracy and hence is useful for building efficient programs.

5. CONCLUSION

In this paper, we derived a policy function error bound (2) for a standard savings problem. The key property of the model that enabled our derivation is the strong concavity of the utility function. The policy error bound holds for any algorithm and is computable for solutions obtained by the fitted value iteration algorithm. Two applications of the error bound are suggested.

APPENDIX A. COMPUTABLE ERROR BOUNDS

Let $\{-\phi = a_1, a_2, \cdots, a_{n-1}, a_n = \bar{a}\} \times \{s_l, s_h\}$ be a partition[2] of X. Define linear interpolation operator L as

$$LV(a,s) = V(a_i,s)\frac{a_{i+1}-a}{a_{i+1}-a_i} + V(a_{i+1},s)\frac{a-a_i}{a_{i+1}-a_i}, \quad a \in [a_i, a_{i+1}].$$

Also, we use V_N to denote the approximate value function obtained at the end of the Nth iteration[3], T to denote the Bellman operator

$$TV(a,s) = \max_{a' \in \Gamma(a,s), a' \leq \bar{a}} u(a+s-qa') + \beta \sum_{s' \in S} V(a',s')\pi(s'|s), \quad (a,s) \in X.$$

and $g_N : X \longrightarrow \mathbb{R}$ to denote the V_N optimal policy function

$$g_N(a,s) \in \text{argmax}_{a' \in \Gamma(a,s), a' \leq \bar{a}} \quad u(a+s-qa') + \beta \sum_{s' \in S} V_N(a',s')\pi(s'|s).$$

Finally, define $e_N := \max_{1 \leq i \leq n, 1 \leq j \leq 2} |V_N(a_i,s_j) - V_{N-1}(a_i,s_j)|$ and $R_N := \max_{1 \leq i \leq n-1, 1 \leq j \leq 2} |TV_N(a_{i+1},s_j) - TV_N(a_i,s_j)|$.

Proposition A.1. *If V_0 is monotone increasing, then*

$$||V^* - V_{g_N}||_\infty \leq \frac{2}{1-\beta}(\beta e_N + R_N).$$

where $||\cdot||_\infty$ denotes the supremum norm.

Proof. By Theorem 4.1 in Stachurski (2008),

$$||V^* - V_{g_N}||_\infty \leq \frac{2}{1-\beta}\left(\beta||V_N - V_{N-1}||_\infty + ||LTV_N - TV_N||_\infty\right). \tag{3}$$

First,

$$\begin{aligned} ||V_N - V_{N-1}||_\infty &= \sup_{(a,s)\in X} \left| [V_N(a_i,s) - V_{N-1}(a_i,s)]\frac{a_{i+1}-a}{a_{i+1}-a_i} \right. \\ &\qquad \left. + [V_N(a_{i+1},s) - V_{N-1}(a_{i+1},s)]\frac{a-a_i}{a_{i+1}-a_i} \right| \\ &\leq e_N. \end{aligned}$$

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[2]This partition requires an upper bound on savings. The existence of such a bound is shown in Huggett (1993). In practice, we can set $\bar{a}$ to a large value.

[3]For details of the fitted value iteration algorithm, see Stachurski (2008).

Secondly, since T and L preserve monotonicity[4] and $V_N = (LT)^N V_0$, TV_N is monotone increasing whenever V_0 is. Using this, we have

$$\begin{aligned} ||LTV_N - TV_N||_\infty &= \sup_{(a,s)\in\bar{X}} \bigg| [TV_N(a_i,s) - TV_N(a,s)]\frac{a_{i+1}-a}{a_{i+1}-a_i} \\ &\qquad + [TV_N(a_{i+1},s) - TV_N(a,s)]\frac{a-a_i}{a_{i+1}-a_i} \bigg| \\ &\leq \sup_{(a,s)\in\bar{X}} |TV_N(a_i,s) - TV_N(a,s)| + |TV_N(a_{i+1},s) - TV_N(a,s)| \\ &= \sup_{(a,s)\in\bar{X}} TV_N(a_{i+1},s) - TV_N(a_i,s) = R_N. \end{aligned}$$

Substitution of e_N and R_N into (3) yields Proposition A.1. □

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[4]An operator $T : W \longrightarrow W$ is said to preserve monotonicity if Tw is increasing for all increasing functions $w \in W$.

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