Partial Information Implementation in Dynare

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1 Introduction

The aim of this document is to describe an algorithm for turning the state space setup of Dynare into one that is suitable for obtaining the partial information setup that conforms to that of Pearlman *et al.* (1986). The state space setup for Dynare is based on writing an RE system as:

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + B u_t \tag{1}$$

where A_0 is not of full rank and u_t is a vector containing instruments w_t and shocks ε_t. Currently estimation within Dynare assumes that agents have full information about the system, so that a calculation is done which solves (1) under full information. The estimation step then assumes that econometricians have only a limited information set, and processes this via the Kalman filter to obtain the likelihood function for a given set of parameters. In reality, agents too have a partial information set (which may or may not coincide with that of the econometricians) given by

$$m_t = L Y_t + v_t \tag{2}$$

where typically there is no observation error ($v_t = 0$) and L picks out most of the economic variables, typically excluding capital stock, Tobin's q and shocks.

The Pearlman *et al.* (1986) setup is given by

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w_t \tag{3}$$

with agents' measurements given by

$$m_t = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + v_t \tag{4}$$

and these can be solved together to yield a reduced-form system. This can then be processed via the Kalman filter to obtain the likelihood function as above.

The next section describes an algorithm for converting the state space (1), (2) under partial information to the form (3), (4).

2 Conversion to Pearlman *et al.* (1986) Setup

In order to reduce the amount of notation we impose a particular way of incorporating shocks into the system. Suppose a particular shock $\bar{m}_t$ affects an equation of the system, where $\bar{m}_{t+1} = \rho\bar{m}_t + \bar{u}_{t+1}$. Redefine $m_t = \bar{m}_{t+1}$, $u_t = \bar{u}_{t+1}$, so that now the relevant equation of the system is affected by m_{t-1}, and the law of motion of the shock is described within the matrices A_1, A_2, B. This makes no difference to the Kalman filter below or to system estimation, but means that for simulation purposes, a shock to u_t at time 0 will have an effect that is diminished by ρ compared with a shock to $\bar{u}_t$ at time 0.

To repeat, all shocks $\bar{m}_t$ to the system at time t are dated as though they were m_{t-1}.

The procedure for conversion to a form suitable for filtering is then as follows:

1. Obtain the singular value decomposition for matrix A_0: $A_0 = UDV^T$, where U, V are unitary matrices. Assuming that only the first m values of the diagonal matrix D are non-zero, we can rewrite this as $A_0 = U_1 D_1 V_1^T$, where U_1 are the first m columns of U, D_1 is the first $m \times m$ block of D and V_1^T are the first m rows of V^T.

2. Multiply (1) by $D_1^{-1}U_1^T$, which yields

$$V_1^T Y_{t+1,t} + D_1^{-1}U_1^T A_1 Y_t = D_1^{-1}U_1^T A_2 Y_{t-1} + D_1^{-1}U_1^T B u_t \tag{5}$$

   Now define $x_t = V_1^T Y_t$, $s_t = V_2^T Y_t$, and use the fact that $I = VV^T = V_1V_1^T + V_2V_2^T$ to rewrite this as:

$$x_{t+1,t} + D_1^{-1}U_1^T A_1(V_1x_t + V_2s_t) = D_1^{-1}U_1^T A_2(V_1x_{t-1} + V_2s_{t-1}) + D_1^{-1}U_1^T B u_t \tag{6}$$

3. Multiply (1) by U_2^T which yields

$$U_2^T A_1 Y_t = U_2^T A_2 Y_{t-1} + U_2^T B u_t \tag{7}$$

   which can be rewritten as

$$U_2^T A_1(V_1x_t + V_2s_t) = U_2^T A_2(V_1x_{t-1} + V_2s_{t-1}) + U_2^T B u_t \tag{8}$$

4. Typically $U_2^T A_1 V_2$ is invertible, which means that we can rewrite (6) and (8) as

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ F & 0 & I \end{bmatrix} \begin{bmatrix} s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & -G_{13} \\ 0 & 0 & I \\ G_{31} & G_{32} & -G_{33} \end{bmatrix} \begin{bmatrix} s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} H_1 \\ 0 \\ H_3 \end{bmatrix} u_t \tag{9}$$

   where

$$G_{11} = (U_2^T A_1 V_1)^{-1} U_2^T A_2 V_2 \quad G_{12} = (U_2^T A_1 V_1)^{-1} U_2^T A_2 V_1 \quad G_{13} = (U_2^T A_1 V_1)^{-1} U_2^T A_1 V_2 \tag{10}$$

$$G_{21} = D_1^{-1}U_1^T A_2 V_2 \quad G_{22} = D_1^{-1}U_1^T A_2 V_1 \quad G_{23} = D_1^{-1}U_1^T A_1 V_2 \tag{11}$$

$$H_1 = (U_2^T A_1 V_1)^{-1}U_2^T B \quad H_3 = D_1^{-1}U_1^T B \quad F = D_1^{-1}U_1^T A_1 V_2 \tag{12}$$

which can be further rewritten as

$$\begin{bmatrix} s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & -G_{13} \\ 0 & 0 & I \\ G_{31} - FG_{11} & G_{32} - FG_{12} & -G_{33} + FG_{13} \end{bmatrix} \begin{bmatrix} s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} H_1 \\ 0 \\ H_3 - FH_1 \end{bmatrix} u_t \tag{13}$$

5. The measurements $m_t = MY_t + v_t$ can be written in terms of the states as $m_t = M(V_1 x_t + V_2 s_t) + v_t$. To write the system in a form which corresponds to that of Pearlman *et al.* (1986) we need to write the measurements in terms of the forward-looking variables x_t and in terms of the backward-looking variables s_{t-1}, x_{t-1}. We do this by substituting for s_t from (13); but this introduces a term in u_t into the expression, and Pearlman *et al.* (1986) assume that shock terms in the dynamics and in the measurements are uncorrelated with one another. To remedy this, we incorporate ε_t into the predetermined variables, but we can retain w_t as it stands.

   Defining

$$\begin{bmatrix} H_1 \\ 0 \\ H_3 - FH_1 \end{bmatrix} u_t = \begin{bmatrix} P_1 \\ 0 \\ P_3 \end{bmatrix} \varepsilon_t + \begin{bmatrix} N_1 \\ 0 \\ N_3 \end{bmatrix} w_t \tag{14}$$

   we may rewrite the dynamics and measurement equations in the form:

$$\begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ P_1 & G_{11} & G_{12} & -G_{13} \\ 0 & 0 & 0 & I \\ P_3 & G_{31} - FG_{11} & G_{32} - FG_{12} & -G_{33} + FG_{13} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & N_1 \\ 0 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1} \\ w_t \end{bmatrix} \tag{15}$$

$$m_t = \begin{bmatrix} LV_2P_1 & LV_2G_{11} & LV_2G_{12} & LV_1 - LV_2G_{13} \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + LV_2N_1w_t + v_t \tag{16}$$

Thus the setup is as required, with the vector of predetermined variables given by $[\varepsilon_t' \; s_{t-1}' \; x_{t-1}']'$, and the vector of jump variables given by x_t. Note that there is an issue not covered by Pearlman (1992), namely that the instrument w_t is part of the measurement equation; if we assume that the instruments are observed, then there is no problem to modify the theory.

There is also a minor issue that the states of the system are not readily identifiable, as they will be linear combinations of the identifiable variables, which may make debugging of errors more problematic.

3 Passing the Model to ACES

The model setup in this form is passed from Dynare to ACES where it is in **Form 2**:

$$\begin{bmatrix} z_{t+1} \\ E_t x_{t+1} \end{bmatrix} = A \begin{bmatrix} z_t \\ x_t \end{bmatrix} + D w_t + \begin{bmatrix} C \\ 0 \end{bmatrix} u_{t+1} \tag{17}$$

$$Y_t + E2 \begin{bmatrix} z_t \\ x_t \end{bmatrix} + E5 w_t = 0 \tag{18}$$

where w_t are the instruments and u_t are the shocks. Note that in ACES notation, $B = I, AB = 0, E1 = I, E4 = 0, E3 = 0$.

For the partial information setup we also require the measurements (2):

$$m_t = L Y_t + v_t \tag{19}$$

N.B. There is one difference here, namely that there is a shock v_t to the measurement Y_t. This shock v_t could also be incorporated into the state vector, by having an additional predetermined variable v_{t+1}. Also Y_t plays a different role here from what it usually does in ACES. In ACES, it represents static relationships that are included in the dynamics, whereas here Y_t represents what is observed by agents and policymakers.

The matrices above then correspond to those of the previous section via:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ P_1 & G_{11} & G_{12} & -G_{13} \\ 0 & 0 & 0 & I \\ P_3 & G_{31} - FG_{11} & G_{32} - FG_{12} & -G_{33} + FG_{13} \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ N_1 \\ 0 \\ N_3 \end{bmatrix} \quad C = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

$$E2 = - \begin{bmatrix} V_2 H_1 & V_2 G_{11} & V_2 G_{12} & V_1 - V_2 G_{13} \end{bmatrix} \qquad E5 = -V_2 N_1 \tag{21}$$

4 Impulse Response Functions

Full Information Case:

It is easy to see that the impulse response functions can be calculated from

$$z_{t+1} = (A_{11} - A_{12} N) z_t + C u_{t+1} \qquad x_t = -N z_t \qquad Y_t = -E2 \begin{bmatrix} z_t \\ x_t \end{bmatrix} \tag{22}$$

where

$$\begin{bmatrix} N & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{23}$$

Partial Information Case: First rewrite m_t as:

$$m_t = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + v_t \tag{24}$$

The reduced-form solution is then given by:

$$\begin{aligned}
System: \quad z_{t+1} &= Fz_t + (A-F)\tilde{z}_t \\
&\quad + (F-A)PH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) + Cu_{t+1} \qquad (25) \\
x_t &= -Nz_t + (N - A_{22}^{-1}A_{21})\tilde{z}_t \\
&\quad -(N - A_{22}^{-1}A_{21})PH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) \qquad (26) \\
Innovations: \quad \tilde{z}_{t+1} &= A\tilde{z}_t - APH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) + Cu_{t+1} \qquad (27) \\
Measurement: \quad m_t &= Ez_t + (H-E)\tilde{z}_t + v_t \\
&\quad -(H-E)PH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) \\
&= Ez_{t,t-1} + (EPH^T+V)(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) \qquad (28)
\end{aligned}$$

where $F = A_{11} - A_{12}N \quad A = A_{11} - A_{12}A_{22}^{-1}A_{21} \quad E = K_1 - K_2N \quad H = K_1 - K_2A_{22}^{-1}A_{21}$

V is the covariance matrix of the measurement errors, and P is the solution of the Riccati equation given by

$$P = APA^T - APH^T(HPH^T+V)^{-1}HPA^T + CUC^T \qquad (29)$$

and U is the covariance matrix of the shocks to the system.

Note that to obtain the impulse response for the underlying variables Y_t we use the relationship

$$Y_t = V_1x_t + V_2s_t \qquad (30)$$

Noting that $s_t = [0 \;\; I \;\; 0]z_{t+1}$, it follows that we may write

$$Y_t = V_1x_t + \begin{bmatrix} 0 & V_2 & 0 \end{bmatrix}\left(Fz_t + (A-F)\tilde{z}_t + (F-A)PH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t)\right) \qquad (31)$$

or more simply

$$Y_t = \begin{bmatrix} 0 & V_2 & V_1 \end{bmatrix} z_{t+1} \qquad (32)$$

4.1 Covariances and Autocovariances for the Partial Information Case

Pearlman *et al.* (1986) show that

$$cov\begin{bmatrix} \tilde{z}_t \\ z_t \end{bmatrix} = \begin{bmatrix} P & P \\ P & P+M \end{bmatrix} \equiv P_0 \qquad (33)$$

where$M$ satisfies

$$M = FMF^T + FPH^T(HPH^T+V)^{-1}HPF^T \qquad (34)$$

If the dimension of the vector Y_t is n, define Ω_0 as the bottom right $n \times n$ matrix of $(P+M)$. Then it follows that

$$cov(Y_t) = [\; V_2 \quad V_1 \;]\Omega_0 \begin{bmatrix} V_2^T \\ V_1^T \end{bmatrix} \equiv R_0 \qquad (35)$$

To calculate the autocovariances, define

$$\Gamma = \begin{bmatrix} A(I - PH^T(HPH^T + V)^{-1}H) & 0 \\ (A - F)(I - PH^T(HPH^T + V)^{-1}H) & F \end{bmatrix} \tag{36}$$

Then the sequence of auto-covariance matrices of Y_t are defined as follows:

$$E\left(\begin{bmatrix} \tilde{z}_{t+k} \\ z_{t+k} \end{bmatrix}, \begin{bmatrix} \tilde{z}_t \\ z_t \end{bmatrix}\right) \equiv P_k = \Gamma^k P_0 = \Gamma P_{k-1} \tag{37}$$

Defining Ω_k as the bottom right $n \times n$ matrix of P_k, it follows that

$$cov(Y_{t+k}, Y_t) = E(Y_{t+k}Y_t^T) = [\ V_2 \quad V_1\ ]\Omega_k \begin{bmatrix} V_2^T \\ V_1^T \end{bmatrix} \equiv R_k \tag{38}$$

These correspond to the matrices **gamma_y** defined at the bottom of page 41 of the Dynare User Guide. These are then use to generate **autocorr**, the autocorrelation functions of the variables. Thus the autocorrelation function of the ith element of Y is given by the sequence $\frac{(R_1)_{ii}}{(R_0)_{ii}}, \frac{(R_2)_{ii}}{(R_0)_{ii}}, \frac{(R_3)_{ii}}{(R_0)_{ii}}, ...$

In addition the **correlation matrix** of the Y_t variables is defined as

$$Corr = \Delta R_0 \Delta^T \text{ where } \Delta = diag(\sqrt{(R_0)_{11}}, \sqrt{(R_0)_{22}}, \sqrt{(R_0)_{33}}, ...) \tag{39}$$

5 Likelihood function calculation

Here we assume that there are no policy instruments w_t and that the system is saddlepath stable.

The Kalman filtering equation is given by

$$z_{t+1,t} = Fz_{t,t-1} + FP_tH^T(EP_tH^T + V)^{-1}e_t \tag{40}$$

where $e_t = m_t - Ez_{t,t-1}$

$$P_{t+1} = AP_tA^T + U - AP_tH^T(HP_tH^T + V)^{-1}HP_tA^T \tag{41}$$

the latter being a time-dependent Ricatti equation.

The period-t likelihood function is standard:

$$2lnL = -\sum lndet(cov(e_t) - \sum e_t^T(cov(e_t))^{-1}e_t \tag{42}$$

where

$$cov(e_t) = (EP_tH^T + V)(HP_tH^T + V)^{-1}(HP_tE^T + V) \tag{43}$$

Following Pearlman *et al.* (1986), the system is initialised at

$$z_{1,0} = 0 \qquad P_1 = P + M \tag{44}$$

where $P$is the steady state of the Riccati equation above, and M is the solution of the Lyapunov equation

$$M = FMF^T + FPH^T(HPH^T + V)^{-1}HPF^T \tag{45}$$

6 Extension to the case of Expectations of Current Variables

Suppose that expectations (or best estimates) of current variables are included in agents' decision-making and measurements. Then a general setup will be of the form

$$A_0 Y_{t+1,t} + A_1 Y_t = A_2 Y_{t-1} + A_3 Y_{t,t} + B u_t \qquad m_t = L Y_t + M Y_{t,t} + v_t \tag{46}$$

To get this into Blanchard-Kahn format, we follow the same procedures as above with $Y_{t,t}$ as a member of the exogenous variables, and end up with a representation of the form

$$\begin{bmatrix} z_{t+1} \\ E_t x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{47}$$

$$m_t = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} R_1 & R_2 \end{bmatrix} \begin{bmatrix} z_{t,t} \\ x_{t,t} \end{bmatrix} + v_t \tag{48}$$

Then all the equations above for filtering, likelihood calculation, IRFs are identical, with the following altered definitions:

$$F = A_{11} + J_{11} - (A_{12} + J_{12})N \qquad E = K_1 + R_1 - (K_2 + R_2)N \tag{49}$$

$$\begin{bmatrix} N & I \end{bmatrix} \begin{bmatrix} A_{11} + J_{11} & A_{12} + J_{12} \\ A_{21} + J_{21} & A_{22} + J_{22} \end{bmatrix} = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{50}$$

References

Pearlman, J., Currie, D., and Levine, P. (1986). Rational Expectations Models with Private Information. *Economic Modelling*, **3**(2), 90–105.