Applying perturbation methods to incomplete market models with exogenous borrowing constraints*

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Feb 4, 2007

Abstract

This paper solves an incomplete market model with an infinite number of agents and exogenous borrowing constraints described in den Haan, Judd and Juillard (2004). We apply the "barrier methods" to convert optimization problem with borrowing constraints as inequalities into a problem with equality constraints, and the converted model is solved by a second-order perturbation method. The simulation results of impulse responses and second moments match the standardized features of incomplete market models. Accuracy of the solution is in a reasonable range but significantly decreases when the economy is near the borrowing limit or moves away from the steady state.

- JEL Classification: C63, C68, C88, F41;
- Key Words: perturbation, barrier method, borrowing constraint, incomplete market, accuracy.

*We appreciate comments from Ken Judd and seminar participants at the 2003 CFS-JEDC conference and at SITE 2004. The views in this paper are solely the responsibility of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or any other person associated with the Federal Reserve System.

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1 Introduction

This paper describes how to use perturbation methods to solve incomplete market models with exogenous borrowing constraints, as described in "Problem C" of the JEDC Numerical Methods Comparison project (den Haan, Judd, and Juillard, 2004). In particular, we solve an infinite agent model with incomplete asset markets where agents trade risk-free one-period bonds only. Agents face borrowing constraints that are exogenously given. There are idiosyncratic shocks as well as aggregate shocks in the endowment process, where both shocks have a continuous support.

Using a perturbation method to solve this model can be a challenge because of the following three properties of the model. First, there is an infinite number of agents in the model. Applying perturbation methods to an infinite agent model demands a novel implementation of the solution method. Second, the existence of exogenous borrowing constraints makes it hard to use numerical methods associated with equality constraints. Third, the incomplete market model is locally nonstationary by nature.

We overcome these problems as follows. First, in order to solve the infinite-agent problem, we calculate the first order conditions of the representative agent under exogenous asset price. We then derive the process of asset price by using the market clearing condition (i.e., the sum of all asset holdings is equal to zero). We use the linear approximation in the derivation of the analytic solution for asset price, since analytically tractable closed form solution is not available for higher order approximations. Finally, we complete the first order conditions of the representative agent in the infinite agent model by using this linear asset price process.[1] Since the analytic solution is not available for this system, we numerically approximate the model using the second-order perturbation method (quadratic approximation). For algorithms, we use Matlab code `gensys2.m` (Sims, 2001; Kim, Kim, Schaumburg, and Sims, 2003).

In order to deal with hard borrowing constraints, we "soften" the borrowing constraints by modifying utility function so that agents are penalized when borrowing moves close to the "barrier" set by the exogenous bound.[2] With this modified utility function, we can convert an optimization problem with inequality constraints (due to hard borrowing constraints) into an optimization problem with only equality constraints, which allows us to use

[1] This is similar in spirit to the idea of bounded rationality, under which only some moments of variables are considered as in Krusell and Smith (1998).

[2] This method is called "barrier method." We thank Ken Judd for suggesting this modification.

standard numerical solution methods.[3] When applying perturbation method to this model, we use a specific perturbation variable for bond holding so that borrowing limit is never hit in the optimal solution.[4] Utility modification has an additional positive feature that it makes the model stationary, which allows us to derive unconditional moments of variables.

The simulation results suggest the second moments and impulse responses of the model match the standard characteristics of the incomplete market models in international macroeconomics.[5] Accuracy tests based on Euler equation errors suggest that approximation errors increase with tighter borrowing constraint and that quadratic approximation generates smaller errors than linear approximation.

The remaining sections consist of the following. In section 2, we introduce the model and explain how we apply barrier methods to incorporate the borrowing constraints. Section 3 discusses the linearized and quadratic solutions. In section 4, we evaluate the performance of the perturbation solution in three categories. First, we derive impulse responses when aggregate and idiosyncratic shocks hit the economy. Then, we check the properties of the model by deriving second moments of key variables including wealth distribution. Finally, for accuracy test, we compare Euler equation errors from linear and quadratic solutions under difference parameter values. Section 5 concludes the paper.

2 Model

We first present the original model with inequality constraints and then modify the utility function of the model to replace the inequality constraints with equality constraints.

[3] Several papers have studied models with borrowing constraints. Some examples with exogenous bound are Huggett (1993), Levine and Zame (2002), and Kubler and Schmedders (2001). Alternatively, others solve the models with endogenously derived bounds on asset holdings, e.g. endogenous solvency constraints as in Alvarez and Jermann (2000) and enforcement constraints as in Kehoe and Perri (2002).

[4] This is analogous to the way that nonnegativity constraint of consumption is incorporated in macro models by approximating the model with respect to log consumption, which guarantees that optimal solution for consumption never takes a negative value.

[5] For example, see Mendoza (1991), Baxter and Crucini (1995), Kollmann (1996, 1998), and Kim, Kim and Levin (2003).

2.1 The Original Model

Each agent i maximizes

$$\max \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} \tag{1}$$

subject to

$$\begin{aligned} c_t^i + q_t b_t^i &= y_t^i a_t + b_{t-1}^i. & (2) \\ b_t^i &\geq -\bar{b} & (3) \end{aligned}$$

where y_t^i is the idiosyncratic shock, a_t is the aggregate shock that is same for each agent, and $\bar{b}$ is the maximum amount that agent i can borrow.

The first-order conditions for each agent i can be described as the two-part Kuhn-Tucker conditions;

$$q_t \left(c_t^i\right)^{-\gamma} \geq \beta \mathrm{E_t} \left(c_{t+1}^i\right)^{-\gamma}, \tag{4}$$

$$\left(b_{t+1}^i + \bar{b}\right) \left(q_t \left(c_t^i\right)^{-\gamma} - \beta \mathrm{E_t} \left(c_{t+1}^i\right)^{-\gamma}\right) = 0, \tag{5}$$

Equilibrium requires the world resource constraint that the sum of bond holdings over all agents is equal to zero.

$$\sum_{i=1}^{\infty} b_t^i = 0. \tag{6}$$

The driving processes for idiosyncratic and aggregate shocks are

$$\begin{aligned} \log(y_t^i) &= \frac{-0.5(1-\rho_y)\sigma_y^2}{(1-\rho_y^2)} + \rho_y \log(y_{t-1}^i) + \sigma_y e_{y,t}, & (7) \\ \log(a_t) &= \frac{-0.5(1-\rho_a)\sigma_a^2}{(1-\rho_a^2)} + \rho_a \log(a_{t-1}) + \sigma_a e_{a,t}, & (8) \end{aligned}$$

where $e_{y,t}$ and $e_{a,t}$ are i.i.d. random variables with a standard Normal distribution.

2.2 A Modified Model

We incorporate borrowing constraint into optimization problem with equality constraints by modifying utility function as follows: Agents are penalized when they borrow or lend from others. The key assumption is that the penalty diverges to infinity as their borrowing approaches the borrowing limit. In particular, we use the following utility function,

$$U(c_t^i, b_t^i) = \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} + \zeta \bar{b} \left[\bar{b} \log\left(b_t^i + \bar{b}\right) - b_t^i\right] \tag{9}$$

The second term in the utility function represents the barrier function and we choose a specific (and rather complicated) form in order to have its first derivative implies the steady state being independent of ζ. Due to the term $\left[\log\left(b_t^i + \bar{b}\right)\right]$ in the utility function, disutility from borrowing significantly increases when agents borrow too much and b_t^i goes towards $-\bar{b}$. Therefore, optimal solution for the borrowing would not exceed this limit. By adjusting the value of ζ, we can adjust how much weight the agents impose on borrowing constraints.[6] In actual simulations, we choose the value of ζ that minimizes the approximation errors.[7]

Each agent maximizes the discounted sum of this modified utility function subject to the original budget constraint (2). That is, we convert an optimization problem with inequality constraints into an optimization problem with equality constraints. Euler equation for each agent i becomes

$$q_t \left(c_t^i\right)^{-\gamma} = \beta \mathrm{E_t} \left(c_{t+1}^i\right)^{-\gamma} - \zeta \left(\frac{b_t^i \bar{b}}{b_t^i + \bar{b}}\right) \tag{10}$$

A by-product of the utility modification method is that the model becomes stationary and the steady state bond holdings can be uniquely determined (zero in this model). Another popular way to make the model stationary is to adopt bond holding costs.[8] The process of achieving stationarity through modification of utility function works exactly the same way as the bond holding cost does. The simulation results (which are not reported in this paper) confirm that both methods produce similar results for impulse responses and second moments.

[6] The solution of this modified model converges to that of the original model as $\zeta \to 0$. This is an application of barrier methods, which are a practical way of solving constrained optimization problems. Penalty methods—which attributes penalty outside the domain of the problem—are also widely used. See Judd (1998) and Luenberger (1973) for more on these two methods.

[7] Definition of approximation errors and their calculation methods are explained later in this paper.

[8] See Kim and Kose (2003) and Schmitt-Grohe and Uribe (2003) for this and alternative methods to make incomplete market models stationary.

3 Solution

In order to solve the infinite agent model, we first solve the representative agent model given exogenous asset price q_t. Then, we derive the solution for asset price by using the market clearing condition (6). Note that we use linear approximation to derive asset price. Using this asset price process, we can complete the first order conditions for the representative agent.

We derive approximate solutions of the modified model by using a perturbation method. To guarantee the variables stay within the sensible range, we use the following variables for perturbation:

$$\begin{aligned}
\hat{c}_t^i &= \log c_t^i, \\
\hat{a}_t &= \log a_t, \\
\hat{y}_t^i &= \log y_t^i, \\
\hat{q}_t &= \log\left(\frac{q_t}{\beta}\right), \\
\hat{b}_t^i &= \bar{b}\log\left(\frac{b_t^i+\bar{b}}{\bar{b}}\right).
\end{aligned}$$

The last transformation implies that b_t^i cannot move below the exogenous bound.[9]

3.1 Linear Asset Pricing Rule

In this section, we derive the closed-form first-order conditions of the model using linear approximation. We solve the linearized model using eigenvalue decomposition. The linearized version of the first order conditions (budget constraint and Euler equation) for each agent i can be expressed in the following linear system (assuming q_t is exogenously given),

$$\begin{bmatrix} 1 & \beta \\ \gamma & 0 \end{bmatrix}\begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \gamma & -\zeta \end{bmatrix}\begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} \hat{y}_t^i + \hat{a}_t \\ -\hat{q}_{t-1} \end{bmatrix}. \tag{11}$$

[9] This transformation satisfies three satisfactory properties of the transformation $\hat{b}_t^i = f\left(b_t^i\right)$. They are $f(0)=0$, $f'(0)=1$, and $f''(0)=-\frac{1}{\bar{b}}$.

It is straightforward to show, as in the Appendix, that the solution for agent i is

$$\frac{\hat{b}_t^i}{\lambda} = \hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t - \sum_{s=0}^{\infty} \beta^s \mathrm{E}_t \left[(1-\beta)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right) + \frac{\hat{q}_{t+s}}{\gamma} \right] \quad (12)$$

$$\frac{\hat{c}_t^i}{\lambda} = \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\left(\hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t\right) + \sum_{s=0}^{\infty} \beta^{s+1} \mathrm{E}_t \left[(1-\beta)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right) + \frac{\hat{q}_{t+s}}{\gamma} \right], \quad (13)$$

where

$$\lambda = \frac{1}{2}\left[\left(1 + \beta^{-1} + \beta^{-1}\gamma^{-1}\zeta\right) - \sqrt{\left(1 + \beta^{-1} + \beta^{-1}\gamma^{-1}\zeta\right)^2 - 4\beta^{-1}}\right].$$

The next step is to derive the solution for asset price using the linearized first order condition (12) and the market clearing condition (6).[10] Then, we have:

$$0 = 0 + \hat{a}_t - \sum_{s=0}^{\infty} \beta^s \mathrm{E}_t \left[(1-\beta)\hat{a}_{t+s+1} + \frac{\hat{q}_{t+s}}{\gamma} \right] \quad (14)$$

whose solution is

$$\hat{q}_t = \gamma \left[\hat{a}_t - \mathrm{E}_t\left(\hat{a}_{t+1}\right)\right] = \gamma\left(1 - \rho_a\right)\hat{a}_t. \quad (15)$$

We use this linear asset price process to complete the representative agent solution in the infinite agent model. Plugging (15) into (12) and (13), we have the following equilibrium solutions:

$$\frac{\hat{b}_t^i}{\lambda} = \hat{b}_{t-1}^i + \hat{y}_t^i - (1-\beta)\sum_{s=0}^{\infty} \beta^s \mathrm{E}_t\left(\hat{y}_{t+s+1}^i\right), \quad (16)$$

$$\frac{\hat{c}_t^i}{\lambda} = \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\left(\hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t\right) + (1-\beta)\sum_{s=0}^{\infty} \beta^{s+1} \mathrm{E}_t\left(\hat{y}_{t+s+1}^i\right) + \beta\hat{a}_t, \quad (17)$$

[10] Second order approximation of $\hat{q}_t$ involves double summation of quadratic and cross product terms and therefore is not analytically tractable.

Using the AR(1) shock processes in (7) and (8), we can derive the linearized closed-form solutions for $\hat{b}_t^i$ and $\hat{c}_t^i$:

$$\frac{\hat{b}_t^i}{\lambda} = \hat{b}_{t-1}^i + \left(\frac{1-\rho_y}{1-\beta\rho_y}\right)\hat{y}_t^i, \tag{18}$$

$$\frac{\hat{c}_t^i}{\lambda} = \left(1-\lambda\beta+\frac{\zeta}{\gamma}\right)\hat{b}_{t-1}^i + \left(\frac{1-\beta}{1-\beta\rho_y}\right)\hat{y}_t^i + \hat{a}_t. \tag{19}$$

3.2 Quadratic Solution

Using the linear asset pricing rule derived in the previous section, we can complete the equation system for the representative agent in the infinite agent model using quadratic approximation;

$$\begin{aligned} 0 &= \hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t^i - \frac{\left(\hat{c}_t^i\right)^2}{2} - \beta\hat{q}_t\hat{b}_t^i + \\ &\quad \frac{\left(\hat{y}_t^i\right)^2}{2} + \frac{\left(\hat{a}_t^i\right)^2}{2} - \hat{c}_t^i - \beta\hat{b}_t^i, \end{aligned} \tag{20}$$

$$\begin{aligned} 0 &= \hat{q}_t + \frac{\left(\hat{q}_t\right)^2}{2} - \gamma\hat{c}_t^i + \frac{\gamma^2\left(\hat{c}_t^i\right)^2}{2} - \gamma\hat{c}_t^i\hat{q}_t \\ &\quad -\mathrm{E}_t\left[-\gamma\hat{c}_{t+1}^i + \frac{\gamma^2\left(\hat{c}_{t+1}^i\right)^2}{2}\right] + \zeta\hat{b}_t^i - \frac{\left(\hat{b}_t^i\right)^2}{\bar{b}}, \end{aligned} \tag{21}$$

$$\hat{y}_t = \frac{-0.5(1-\rho_y)\sigma_y^2}{(1-\rho_y^2)} + \rho_y\hat{y}_{t-1} + \sigma_y e_{y,t}, \tag{22}$$

$$\hat{a}_t = \frac{-0.5(1-\rho_a)\sigma_a^2}{(1-\rho_a^2)} + \rho_a\hat{a}_{t-1} + \sigma_a e_{a,t} \tag{23}$$

$$\hat{q}_t = \gamma\left(1-\rho_a\right)\hat{a}_t. \tag{24}$$

Since this system of equations cannot be solved analytically, we use numerical approximations for quadratic solutions. We solve the five equation system above using Matlab algorithm `gensys2.m`. In actual simulations, we use the "pruning" algorithm to avoid accumulation of high-order terms that can make the model unstable (Kim, Kim, Schaumburg, and Sims 2003). This pruning algorithm always produces stationary second-order accurate dynamics whenever the first-order dynamics are stable.

4 Model Performance

Parameter values that we use for simulation are summarized in the following table. The size of the idiosyncratic shock is estimated from the PSID data and quite large ($\sigma_y = 0.21$) compared to that is used in macro models (normally around 0.01). Implication of shock volatility on accuracy of the approximate solution is discussed later in this paper.

Parameter	Values
β	0.965
γ	1, 5
$\bar{b}$	0.2, 1
ρ_y	0.49
ρ_a	0.91
σ_y	0.21
σ_a	0.01
ζ	various values

4.1 Policy function

One advantage of the `gensys2.m` algorithm is that one does not have to distinguish state and control variables in solving the model. In particular, this "state-free" approach becomes useful when one solves complicated models with a large number of state variables. When one wants to have decomposition into traditional state and control variables (for example, deriving so-called policy function), `gensys2.m` does not readily provide this decomposition but one can achieve this decomposition using another Matlab algorithm called `gstate.m`.[11]

In this section, we report policy functions in the case with $\bar{b} = 1$, $\gamma = 1$, and $\zeta = 0.05$. Note that we choose the value of ζ that minimizes average approximation errors. In this case, the policy function for bond holdings

[11] See Kim, Kim, Schaumburg and Sims (2003) for details. All algorithms are publicly available at http://sims.princeton.edu/yftp/gensys2.

and consumption are

$$
\begin{aligned}
b_t \;=\;& 0.0032 + 0.8089 b_{t-1} + 0.3274 y_{t-1} \\
& + 0.6681 e_{yt} + 0.4242\,(e_{yt} e_{at}) + 0.1832 e_{yt}^2 \\
& + 0.044 y_{t-1}^2 + 0.1059 b_{t-1}^2 - 0.2397\,(y_{t-1} b_{t-1}) - 0.1983\,(a_{t-1} b_{t-1}) \\
& + 0.1891\,(y_{t-1} a_{t-1}) + 0.2079\,(y_{t-1} e_{at}) - 0.2179\,(b_{t-1} e_{at}) \\
& + 0.1795\,(y_{t-1} e_{yt}) - 0.4892\,(b_{t-1} e_{yt}) + 0.3860\,(a_{t-1} e_{yt}), \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
c_t \;=\;& -0.0004 + 0.2712 b_t + 0.0853 y_{t-1} + 0.91 a_{t-1} \\
& + 0.1741 e_{yt} + e_{at} + 0.0791\,(e_{yt} e_{at}) + 0.0108 e_{yt}^2 \\
& + 0.0026 y_{t-1}^2 + 0.0449 b_t^2 - 0.0262\,(y_{t-1} b_t) - 0.0228\,(a_{t-1} b_t) \\
& + 0.0353\,(y_{t-1} a_{t-1}) + 0.0388\,(y_{t-1} e_{at}) - 0.025\,(b_t e_{at}) \\
& + 0.0106\,(y_{t-1} e_{yt}) - 0.0534\,(b_t e_{yt}) + 0.0720\,(a_{t-1} e_{yt}). \qquad (26)
\end{aligned}
$$

4.2 Impulse responses

In this section, we report impulse responses of key variables to various shocks to the economy. We assume that the economy starts from the deterministic steady state (bond holdings of the representative agent at the steady state are set to zero).

4.2.1 Aggregate shock

In tables 1 and 2, we report impulse responses of consumption, bond holding and bond price (q_t) to one standard deviation shock (positive and negative) to aggregate and idiosyncratic endowments. We compare solutions of linear and quadratic approximations. We experiment with three values for ζ (0, 0.001 and 0.1), two values for γ (1 and 5), and two values of $\bar{b}$ (0.2 and 1).

Table 1 presents the impulse responses to positive and negative aggregate shocks (1 percent) to endowment a_t. Since all agents receive same shocks, there is no bond trading and the economy simply behaves like autarky. Impulse responses reflect this behavior and show that bond holdings stay at zero all the time. Both linear and quadratic approximations produce exactly the same results because asset pricing rule is linear (the quadratic approximation of a linear function is still linear) and consumption is a linear function of asset price. Since bond holdings stay at zero, consumption moves exactly same as output (as in autarky). Asset price increases with positive shocks because increased endowment creates excess supply that pushes interest rate down (asset price up). Note that asset price is a positive function of aggregate endowment as shown in equation (15). The table also shows that changes in γ affects bond prices only: q_t becomes more sensitive to endowment shocks when γ increases. Since bond holdings do not change, the model produce the same impulse responses irrespective of the values of parameters ζ and $\bar{b}$.

4.2.2 Idiosyncratic shock

Table 2 presents the impulse responses to positive and negative shocks to idiosyncratic components of endowment y_t. The first panel in table 2 reports the case of a positive shock; individual endowment increases by 21 percent in the first period and eventually dies out in 30 periods with persistence parameter of 0.49. Since asset price q_t is a function of aggregate shock only, q_t does not respond to the idiosyncratic shock.

Results show that impulse responses present the standard characteristics of typical incomplete markets models as shown in Kollmann (1996, 1998) and Kim,Kim and Levin (2003). With a positive shock, agents save a part of increased income by accumulating bonds over time. When $\zeta = 0$ (no utility modification), there is a permanent effect of temporary shock on consumption (nonstationary property of the incomplete market model). However, with positive ζ, both consumption and bond holding processes are stationary by reacting more to shocks at the initial period and eventually moving back to the original steady state. Changing the value of γ affects the impulse responses when ζ is positive but there is no effect when when $\zeta = 0$. We observe more consumption smoothing with higher γ. Comparison of the linear and quadratic solutions shows that linear approximation underestimates the responses of consumption compared to the quadratic approximation. As ζ increases, differences between linear and quadratic solutions increase.

The second panel in table 2 presents the case with a negative shock. With a negative shock, agents borrow (minus b_t) to smooth out consumption over time. In the case of a linear solution, the response of consumption is exactly the opposite to that of the positive shock case. However, the existence of borrowing limits affects optimal behavior of bond holdings. The absolute amount of borrowing when agents face negative shock is less than the absolute amount of bond accumulation when agents face the same magnitude of positive shock. For example, with positive shock ($\gamma = 1, \zeta = 0$), bond holdings increase by 22% at impact, while with the same magnitude of negative shock, agents' borrowings increase only by 18%. This phenomenon becomes more significant when borrowing limit ($\bar{b}$) is decreased to 0.2. Panel 3 of table 2 shows how bond holdings react when $\bar{b} = 0.2$ compared to the case when $\bar{b} = 1$. With the same size of negative shock, agents do not borrow as much as in the case when $\bar{b} = 1$. For example, borrowing in the first period becomes only 13% instead of 18% when $\bar{b}$ decreases from 1 to 0.2 (with $\gamma = 1, \zeta = 0$). Consumption process remains unchanged with changes in $\bar{b}$.

4.3 Moments

In this section, we report main properties of simulated time series of variables when both aggregate and idiosyncratic shocks hit the economy. All the statistics are based on 500 simulations where each simulation lasts for 100 periods. Since the moments are sensitive to some parameter values, we experiment with three values for ζ (0.1, 0.001 and 10^{-10}), two values of γ (1 and 5), and two values of $\bar{b}$ (1 and 0.2).[12] Actual shocks are generated by random number generator in Matlab with the characteristics reported in the model parameter table. We report correlation between consumption and endowment shocks (aggregate and idiosyncratic), and between consumption and bond holdings. We also report standard deviation and autocorrelation parameters (up to three lags) of consumption, bond holdings and asset price. We report statistics generated by linear and quadratic solutions.

As in the previous section, the statistics are consistent with standard properties of incomplete market models. Consumption is procyclical and less volatile than output (standard deviations of aggregate and idiosyncratic shocks are 0.01 and 0.21, respectively). Standard deviation of consumption is around $5\% \sim 15\%$ depending mostly on the value of ζ. When ζ increases, consumption volatility increases because consumption moves more similarly to the changes in output (less degree of consumption smoothing). Correlation between consumption and output shocks are positive around $0.2 \sim 0.8$, depending on the parameter values of γ and ζ. Consumption and bond holdings are also positively correlated. As ζ increases, consumption correlation with idiosyncratic shock and bond holdings increase because of less amount of consumption smoothing. When ζ is low, consumption process becomes near nonstationary while output process remains stationary, which lowers correlation between consumption and idiosyncratic shock. On the other hand, correlation between consumption and aggregate shock decreases as ζ increases. Both consumption and bond holding processes are quite persistent with autocorrelation parameter with lag 1 are around $0.7 \sim 0.9$. Changes in $\bar{b}$ does not affect consumption process but they affect bond holdings. Bond

[12] We so not use zero for ζ because when ζ is zero, second moments are not well defined due to nonstationarity.

holdings become less persistent and less correlated with consumption with tighter borrowing constraint (low $\bar{b}$).

Bond price q_t is quite persistent with autocorrelation parameter is above 0.85 with lag 1. Process of q_t does not depend on any parameters (ζ nor $\bar{b}$) except that the volatility of bond price is affected by γ; when γ increases from 1 to 5, standard deviation of q_t decreases to a half. Comparing linear and quadratic solutions reveal that both solutions produce similar statistics except that linear solution slightly underestimates consumption volatility. Since bond price q_t is a linear process, both linear and quadratic solutions produce same statistics for q_t.

4.4 Wealth distribution

In this section, we analyze wealth distribution predicted by the model using simulated series of bond holdings. We simulate the economy 1000 times with 100 periods in each simulation. We experiment with two values of $\bar{b}$ (0.2 and 1) while $\gamma = 1$ and $\zeta = 0.001$.[13] We report the results from quadratic solution for this exercise. All simulations start from the deterministic steady state with zero bond holdings.

Since we incorporate borrowing constraint in the way that the constraints never bind, bond holdings never hit the limit. However, bond holdings can get very close to the limit. We derive the fraction of times that agents are constrained by borrowing limits by approximating how often b_t moves close to the borrowing limit. Table 4 reports the results. When $\bar{b}$ is 1, bond holdings move within 10^{-5} of borrowing limit (it is -0.99999) only less than 1% of times. About 44% of times, agents end up in a borrowing situation. In 0.02% of times, agents accumulate bond holdings over 10, which means that agents' net assets become ten times more than the steady state level of consumption..

However, with a tighter borrowing constraint $\bar{b} = 0.2$, agents hit the borrowing constraint around 50% of times (when using the same definition of 10^{-5} range). Wealth distribution has a thicker right tail when $\bar{b} = 0.2$ as agents accumulate bonds over 10 in about 8% of times.

[13]Wealth distribution is quite sensitive to the value of ζ. With a higher value of ζ,the model becomes more stationary. Therefore, wealth distribution would be more concentrated around the steady state and agents would hit the borrowing limit less frequently.

4.5 Accuracy Tests

We evaluate the accuracy of linear and quadratic solutions using Euler equation errors. In order to deal with the possible accumulation of Euler equation errors, we proceed as follows. First, we simulate the model economy for 50 times with 500 period for each simulation and calculate the consumption process using numerical solution. Second, with the same realization of shocks, we derive another consumption series calculated explicitly from the Euler equation. In particular, we take conditional expectation of c_{t+1} in Euler equation at time t using two-dimensional Gauss-Hermite quadrature with ten nodes.[14] Finally, we calculate approximation errors defined as percentage differences of this consumption series from the actual consumption series from our numerical solution. Since the original utility function does not contain penalty terms, it would be appropriate to use the original Euler equation without utility modification when calculating approximation errors.

Table 5 reports mean and maximum absolute approximation errors. Errors are expressed as percentage of steady state consumption. We report errors in selected periods [10, 20, 50, 100, 200, 500] in order to detect possible accumulation of errors. We experiment with two values for borrowing limit $\bar{b}$ (0.2 and 1). The value of ζ is set to minimize approximation errors for each solution method and each set of parameter values.

The first panel is the case when $\bar{b} = 1$. Average approximation errors are minimized when $\zeta = 0.05$. Euler equation errors are between 1.3% and 1.8% of steady state consumption in both linear and quadratic approximations. The quadratic solution generates noticeably smaller average approximation errors than the linear solution. With $\bar{b} = 0.2$, approximation errors increase to $4 \sim 5.5$ percent of steady state consumption. With tighter borrowing constraints (that is, smaller $\bar{b}$), the relative performance of quadratic approximation over linear approximation improves. The average approximation errors of quadratic solution are as much as one percentage point smaller than those of linear solution, while the difference was between $0.2 \sim 0.5\%$ with $\bar{b} = 1$. The superiority of quadratic solution over linear solution becomes more apparent in maximum approximation errors; maximum errors in quadratic solution is less than the half of maximum errors of the linear solution at 200^{th} and 500^{th} periods.

Accuracy of perturbation method is directly related to the volatility of underlying shocks of the economy. The current specification of the model

[14] See Judd (1998) for details. Note that numerical solution is still used to calculate the outcome of the variables (bond holdings) inside the conditional expectation.

uses highly volatile idiosyncratic shocks (standard deviation of 21%). If we lower the shock variance, then the approximation errors would significantly go down. In fact, we experimented with a smaller idiosyncratic shock with $\sigma_y = 0.021$ (one tenth of the original σ). Approximation errors decrease to around 0.1% which is a more-than-proportional improvement compared to the original case (errors were round $1.3 \sim 1.8\%$ when $\sigma_y = 0.21$).

5 Conclusion

This paper explains how to use a perturbation method to solve an incomplete market model. Traditionally, perturbation method has not been used to solve this type of model because perturbation method is a local approximation and deals mostly with equality constraints. We adopt the barrier method to convert the maximization problem with inequality constraints into an equality constraint problem. Simulation results show that the perturbation solution generates quite reasonable results in terms of second moments and impulse responses. Accuracy of the solution is in a reasonable range. Considering the computation time and the easiness of applying the solution, perturbation methods deserve some consideration compared to other computationally intensive projection solution methods.

A Derivation of the Linear Solution

The linearized version of the first order conditions (budget constraint and Euler equation) for each agent (country) i can be expressed in the following linear system (expectation operator is omitted for convenience),

$$\begin{bmatrix} 1 & \beta \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \gamma & -\zeta \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} \hat{y}_t^i + \hat{a}_t \\ -\hat{q}_{t-1} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & \beta \\ \gamma & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\beta} & -\frac{1}{\beta\gamma} \end{bmatrix}$$

the system becomes

$$\begin{aligned} \begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\beta} & -\frac{1}{\beta\gamma} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \gamma & -\zeta \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\beta} & -\frac{1}{\beta\gamma} \end{bmatrix} \begin{bmatrix} \hat{y}_t^i + \hat{a}_t \\ -\hat{q}_{t-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{\zeta}{\gamma} \\ -\frac{1}{\beta} & \frac{1}{\beta}\left(1 + \frac{\zeta}{\gamma}\right) \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix} + \begin{bmatrix} -\frac{\hat{q}_{t-1}}{\gamma} \\ \frac{1}{\beta}\left(\hat{y}_t^i + \hat{a}_t + \frac{\hat{q}_{t-1}}{\gamma}\right) \end{bmatrix} \end{aligned}$$

Based on the Jordan decomposition

$$\begin{aligned} &\begin{bmatrix} 1 & -\frac{\zeta}{\gamma} \\ -\frac{1}{\beta} & \frac{1}{\beta}\left(1 + \frac{\zeta}{\gamma}\right) \end{bmatrix} \\ = &\begin{bmatrix} \left(\Delta + \frac{\zeta}{\gamma}\right) & -\frac{\beta}{1-\beta}\frac{\zeta}{\gamma} \\ 1 & \frac{\Delta}{(1-\beta)\lambda} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda\beta} \end{bmatrix} \begin{bmatrix} \frac{\Delta}{\Delta^2 + \frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2 + \frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2 + \frac{\zeta}{\gamma}} & \frac{\left(\Delta + \frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2 + \frac{\zeta}{\gamma}} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} \lambda &= \frac{1}{2}\left[\left(1 + \beta^{-1} + \beta^{-1}\gamma^{-1}\zeta\right) - \sqrt{\left(1 + \beta^{-1} + \beta^{-1}\gamma^{-1}\zeta\right)^2 - 4\beta^{-1}}\right], \\ \Delta &= 1 - \lambda\beta \end{aligned}$$

we convert the system

$$\begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix}$$

$$= \begin{bmatrix} \left(\Delta + \frac{\zeta}{\gamma}\right) & -\frac{\beta}{1-\beta}\frac{\zeta}{\gamma} \\ 1 & \frac{\Delta}{(1-\beta)\lambda} \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda\beta} \end{bmatrix} \begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{\hat{q}_{t-1}}{\gamma} \\ \frac{1}{\beta}\left(\hat{y}_t^i + \hat{a}_t + \frac{\hat{q}_{t-1}}{\gamma}\right) \end{bmatrix}$$

that is

$$\begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} \hat{c}_t^i \\ \hat{b}_t^i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} \hat{c}_{t-1}^i \\ \hat{b}_{t-1}^i \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\Delta}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\beta\lambda\frac{\zeta}{\gamma}}{\Delta^2+\frac{\zeta}{\gamma}} \\ -\frac{(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} & \frac{\left(\Delta+\frac{\zeta}{\gamma}\right)(1-\beta)\lambda}{\Delta^2+\frac{\zeta}{\gamma}} \end{bmatrix} \begin{bmatrix} -\frac{\hat{q}_{t-1}}{\gamma} \\ \frac{1}{\beta}\left(\hat{y}_t^i + \hat{a}_t + \frac{\hat{q}_{t-1}}{\gamma}\right) \end{bmatrix}$$

To suppress the diverging eigenvalue, we should have

$$-\hat{c}_t^i + \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\hat{b}_t^i = -\sum_{s=0}^{\infty}\beta^s \mathrm{E}_t\left[\frac{\hat{q}_{t+s}}{\gamma} + \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right)\right].$$

Therefore, the solution for agent i given exogenous interest rate becomes

$$\begin{aligned} \frac{\hat{b}_t^i}{\lambda} &= \hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t - \sum_{s=0}^{\infty}\beta^s \mathrm{E}_t\left[(1-\beta)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right) + \frac{\hat{q}_{t+s}}{\gamma}\right] \\ \frac{\hat{c}_t^i}{\lambda} &= \left(1 - \lambda\beta + \frac{\zeta}{\gamma}\right)\left(\hat{b}_{t-1}^i + \hat{y}_t^i + \hat{a}_t\right) \\ &\quad + \sum_{s=0}^{\infty}\beta^{s+1}\mathrm{E}_t\left[(1-\beta)\left(\hat{y}_{t+s+1}^i + \hat{a}_{t+s+1}\right) + \frac{\hat{q}_{t+s}}{\gamma}\right] \end{aligned}$$

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Table 1. Impulse responses to aggregate shock

γ=1 (ζ=0, 0.001, 0.1 and bbar=0.2, 1)

	positive shock (1%)			negative shock (-1%)		
period	c_{t}	b_{t}	q_{t}	c_{t}	b_{t}	q_{t}
1	0.01	0	0.0009	-0.01	0	-0.0009
2	0.0091	0	0.0008	-0.0091	0	-0.0008
3	0.0083	0	0.0007	-0.0083	0	-0.0007
10	0.0043	0	0.0004	-0.0043	0	-0.0004
30	0.0007	0	0.0001	-0.0007	0	-0.0001

γ=5 (ζ=0, 0.001, 0.1 and bbar=0.2, 1)

	positive shock (1%)			negative shock (-1%)		
period	c_{t}	b_{t}	q_{t}	c_{t}	b_{t}	q_{t}
1	0.01	0	0.0045	-0.01	0	-0.0045
2	0.0091	0	0.0041	-0.0091	0	-0.0041
3	0.0083	0	0.0037	-0.0083	0	-0.0037
10	0.0043	0	0.0019	-0.0043	0	-0.0019
30	0.0007	0	0.0003	-0.0007	0	-0.0003

Table 2. Impulse responses to idiosyncratic shock (one standard deviation)

Positive shock (21%) with bbar=1

	y_{t}	c_{t}		b_{t}		q_{t}
		linear	quad	linear	quad	
(γ=1,ζ=0)						
1	0.21	0.0139	0.0149	0.2253	0.2267	0
2	0.1029	0.0139	0.0149	0.3535	0.3284	0
3	0.0504	0.0139	0.0149	0.4212	0.3744	0
10	0.0003	0.0139	0.0149	0.4889	0.4156	0
30	0	0.0139	0.0149	0.4894	0.4158	0
(γ=1,ζ=0.001)						
1	0.21	0.0209	0.0209	0.2165	0.2206	0
2	0.1029	0.0207	0.0207	0.3341	0.317	0
3	0.0504	0.0204	0.0205	0.3907	0.3582	0
10	0.0003	0.0179	0.0186	0.3896	0.3637	0
30	0	0.0124	0.0138	0.2567	0.27	0
(γ=1,ζ=0.1)						
1	0.21	0.0932	0.0968	0.1286	0.137	0
2	0.1029	0.0807	0.0832	0.1599	0.1644	0
3	0.0504	0.0653	0.0674	0.1483	0.1515	0
10	0.0003	0.009	0.0096	0.0229	0.0245	0
30	0	0	0	0.0001	0.0001	0
(γ=5,ζ=0.001)						
1	0.21	0.0159	0.0164	0.2229	0.2252	0
2	0.1029	0.0158	0.0164	0.3482	0.3255	0
3	0.0504	0.0157	0.0163	0.4127	0.3703	0
10	0.0003	0.0152	0.0159	0.4595	0.4027	0
30	0	0.0138	0.0148	0.4092	0.3776	0
(γ=5,ζ=0.1)						
1	0.21	0.0544	0.0579	0.175	0.1805	0
2	0.1029	0.051	0.0538	0.2471	0.2411	0
3	0.0504	0.0465	0.0487	0.2623	0.251	0
10	0.0003	0.0197	0.0208	0.1211	0.1224	0
30	0	0.0017	0.0018	0.01	0.0107	0

Negative shock (-21%) with bbar=1

	y_{t}	c_{t}		b_{t}		q_{t}
		linear	quad	linear	quad	
(γ=1,ζ=0)						
1	-0.21	-0.0139	-0.013	-0.1839	-0.1829	0
2	-0.1029	-0.0139	-0.013	-0.2612	-0.2749	0
3	-0.0504	-0.0139	-0.013	-0.2964	-0.3195	0
10	-0.0003	-0.0139	-0.013	-0.3284	-0.3615	0
30	0	-0.0139	-0.013	-0.3286	-0.3617	0
(γ=1,ζ=0.001)						
1	-0.21	-0.0209	-0.0209	-0.1779	-0.1751	0
2	-0.1029	-0.0207	-0.0207	-0.2504	-0.26	0
3	-0.0504	-0.0204	-0.0203	-0.2809	-0.2977	0
10	-0.0003	-0.0179	-0.0173	-0.2803	-0.2937	0
30	0	-0.0124	-0.0111	-0.2043	-0.1959	0
(γ=1,ζ=0.1)						
1	-0.21	-0.0932	-0.0897	-0.114	-0.1074	0
2	-0.1029	-0.0807	-0.0783	-0.1379	-0.1346	0
3	-0.0504	-0.0653	-0.0633	-0.1292	-0.1268	0
10	-0.0003	-0.009	-0.0083	-0.0224	-0.0208	0
30	0	0	0	-0.0001	-0.0001	0
(γ=5,ζ=0.001)						
1	-0.21	-0.0159	-0.0153	-0.1822	-0.1807	0
2	-0.1029	-0.0158	-0.0152	-0.2582	-0.2707	0
3	-0.0504	-0.0157	-0.0152	-0.2921	-0.3134	0
10	-0.0003	-0.0152	-0.0145	-0.3149	-0.3415	0
30	0	-0.0138	-0.0128	-0.2904	-0.3063	0
(γ=5,ζ=0.1)						
1	-0.21	-0.0544	-0.0509	-0.1489	-0.1449	0
2	-0.1029	-0.051	-0.0483	-0.1982	-0.202	0
3	-0.0504	-0.0465	-0.0442	-0.2078	-0.2149	0
10	-0.0003	-0.0197	-0.0186	-0.108	-0.107	0
30	0	-0.0017	-0.0016	-0.0099	-0.0093	0

Changes in b_{t} when bbar=0.2

(only b_{t} changes when bbar changes)

	b_{t} (positive shock)		b_{t} (negative shock)	
	linear	quad	linear	quad
(γ=1,ζ=0)				
1	0.3523	0.1677	-0.1276	-0.1518
2	0.7086	0.1308	-0.156	-0.184
3	0.9596	0.0851	-0.1655	-0.1915
10	1.2635	0.0331	-0.1727	-0.1956
30	1.2658	0.0327	-0.1727	-0.1957
(γ=1,ζ=0.001)				
1	0.3328	0.1805	-0.1249	-0.1464
2	0.6451	0.1669	-0.1527	-0.1795
3	0.8403	0.1411	-0.1616	-0.1874
10	0.8361	0.2153	-0.1614	-0.1845
30	0.427	0.4163	-0.1362	-0.1373
(γ=1,ζ=0.1)				
1	0.1662	0.1534	-0.0908	-0.0946
2	0.22	0.19	-0.1048	-0.1115
3	0.1994	0.188	-0.0998	-0.1027
10	0.024	0.0338	-0.0214	-0.0136
30	0.0001	0.0001	-0.0001	0
(γ=5,ζ=0.001)				
1	0.3469	0.1727	-0.1269	-0.1502
2	0.6907	0.1435	-0.1551	-0.1827
3	0.9253	0.1038	-0.1645	-0.1904
10	1.1247	0.0929	-0.1698	-0.1933
30	0.9115	0.2138	-0.164	-0.1866
(γ=5,ζ=0.1)				
1	0.2479	0.1809	-0.1107	-0.1241
2	0.4034	0.217	-0.1337	-0.1542
3	0.441	0.2327	-0.1376	-0.1579
10	0.1542	0.1901	-0.0871	-0.0756
30	0.0103	0.0166	-0.0098	-0.004

Table 3. Moments

bbar	1						0.2					
γ	1			5			1			5		
ζ	10^(-10)	0.001	0.1	10^(-10)	0.001	0.1	10^(-10)	0.001	0.1	10^(-10)	0.001	0.1
linear												
corr(C,A)	0.3437	0.2951	0.1421	0.3437	0.3302	0.1867	0.3437	0.2951	0.1421	0.3437	0.3302	0.1867
corr(C,Y)	0.2241	0.373	0.8531	0.2241	0.2707	0.69	0.2241	0.373	0.8531	0.2241	0.2707	0.69
corr(C,B)	0.7588	0.8214	0.9578	0.7588	0.781	0.9251	0.4156	0.4637	0.7861	0.4156	0.4327	0.6148
s.d.(C)	0.0579	0.0665	0.1558	0.0579	0.0595	0.1141	0.0579	0.0665	0.1558	0.0579	0.0595	0.1141
autocorr(C)-1	0.9184	0.9076	0.7712	0.9184	0.9152	0.8518	0.9184	0.9076	0.7712	0.9184	0.9152	0.8518
-2	0.8434	0.8217	0.5661	0.8434	0.8371	0.7126	0.8434	0.8217	0.5661	0.8434	0.8371	0.7126
-3	0.7735	0.742	0.4013	0.7735	0.7643	0.5889	0.7735	0.742	0.4013	0.7735	0.7643	0.5889
autocorr(B)-1	0.9261	0.9323	0.8681	0.9261	0.9294	0.9164	0.6543	0.6807	0.7561	0.6543	0.6666	0.728
-2	0.8279	0.8322	0.6726	0.8279	0.8313	0.7829	0.3843	0.4063	0.4865	0.3843	0.3935	0.4505
-3	0.7287	0.728	0.4876	0.7287	0.7308	0.6447	0.2296	0.2418	0.2948	0.2296	0.2358	0.2716
mean(q)	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654
s.d.(q)	0.0018	0.0018	0.0018	0.009	0.009	0.009	0.0018	0.0018	0.0018	0.009	0.009	0.009
autocorr(q)-1	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567
-2	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327
-3	0.6231	0.6231	0.6231	0.623	0.623	0.623	0.6231	0.6231	0.6231	0.623	0.623	0.623
quad												
corr(C,A)	0.3215	0.2876	0.1417	0.3329	0.3169	0.1858	0.3215	0.2876	0.1417	0.3329	0.3169	0.1858
corr(C,Y)	0.2149	0.3702	0.8592	0.2104	0.2577	0.6958	0.2149	0.3702	0.8592	0.2104	0.2577	0.6958
corr(C,B)	0.3555	0.5039	0.961	0.3395	0.3843	0.9287	0.0834	0.1022	0.5935	0.073	0.0802	0.2812
s.d.(C)	0.0676	0.0717	0.1586	0.069	0.0677	0.1128	0.0676	0.0717	0.1586	0.069	0.0677	0.1128
autocorr(C)-1	0.9217	0.9083	0.7681	0.922	0.9186	0.8509	0.9217	0.9083	0.7681	0.922	0.9186	0.8509
-2	0.8498	0.8237	0.561	0.8503	0.8437	0.7105	0.8498	0.8237	0.561	0.8503	0.8437	0.7105
-3	0.7838	0.7455	0.3957	0.7844	0.7747	0.5857	0.7838	0.7455	0.3957	0.7844	0.7747	0.5857
autocorr(B)-1	0.9275	0.9357	0.8682	0.9275	0.9306	0.9196	0.5781	0.6376	0.7264	0.5786	0.6039	0.6895
-2	0.8312	0.8398	0.6734	0.831	0.8355	0.7898	0.3188	0.3822	0.4405	0.3172	0.3398	0.4119
-3	0.7368	0.7398	0.4892	0.7366	0.7413	0.6537	0.2073	0.254	0.2588	0.2058	0.2246	0.256
mean(q)	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654	0.9651	0.9651	0.9651	0.9654	0.9654	0.9654
s.d.(q)	0.0018	0.0018	0.0018	0.009	0.009	0.009	0.0018	0.0018	0.0018	0.009	0.009	0.009
autocorr(q)-1	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567	0.8567
-2	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327	0.7327
-3	0.6231	0.6231	0.6231	0.623	0.623	0.623	0.6231	0.6231	0.6231	0.623	0.623	0.623

Table 4. Wealth Distribution

	bbar=1	bbar=0.2
	fraction	fraction
within 10^(-6)	0.41%	46.52%
within 10^(-5)	0.83%	49.57%
within 10^(-4)	1.75%	53.20%
within 10^(-3)	3.60%	57.45%
within 10^(-2)	7.35%	62.65%
within 0.1	16.07%	69.71%
b<0	43.39%	72.72%
b<5	98.47%	89.36%
b<10	99.98%	91.94%

percentile	b	b
5%	-0.997	-0.2
10%	-0.9747	-0.2
90%	2.8933	5.9014
95%	2.8994	5.9749

Table 5. Accuracy tests (Euler equation errors)
(50 agents with 500 period each)

bbar=1 ζ=0.05

	Linear		**Quad**	
period	mean	*max*	mean	*max*
10	1.66%	*6.40%*	1.70%	*5.64%*
20	1.64%	*4.08%*	1.63%	*5.59%*
30	1.81%	*10.90%*	1.66%	*7.66%*
50	1.28%	*4.28%*	1.38%	*5.22%*
100	1.87%	*8.08%*	1.29%	*5.09%*
200	1.64%	*8.45%*	1.35%	*5.25%*
500	1.82%	*10.11%*	1.57%	*6.46%*

bbar=0.2 ζ=0.5

	Linear		**Quad**	
period	mean	*max*	mean	*max*
10	4.66%	*20.42%*	4.17%	*13.55%*
20	4.53%	*15.00%*	4.10%	*11.36%*
30	5.00%	*19.62%*	4.66%	*16.21%*
50	3.83%	*11.18%*	3.42%	*7.35%*
100	5.03%	*18.41%*	4.09%	*12.03%*
200	5.21%	*28.38%*	4.37%	*14.73%*
500	5.59%	*40.98%*	4.77%	*19.15%*