SOLVING HETEROGENEOUS-AGENT MODELS WITH PARAMETERIZED CROSS-SECTIONAL DISTRIBUTIONS

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Key issue in dynamic heterogeneous agent models. How to approximate the law of motion of the wealth distribution?

- Key trick (Den Haan 1996,1997, Krusell and Smith 1997,1998, Rios-Rull 1997):
  - Summarize cross-sectional distribution with a set of moments
  - Express next period's moments as a function of current period ones and aggregate shocks

A: Traditional methods

Most popular one: Krusell and Smith 1998

1. Parameterize the law of motion for moments

$$m_{t+1} = P_n(a_{t+1}, a_t, m_t | \phi_{m,n})$$

2. Solve individual policy rules with your favorite algorithm

$$k^i_{t+1} = P_n(k^i_t, \varepsilon_t, a_t, m_t | \phi_{z,n})$$

3. Use simulations (Monte-Carlo) to do the numerical integration and up-date the law of motion

A: Traditional methods

Most popular one : Krusell and Smith 1998

- **Pro: simulation**
  - Tractability
  - No restriction on functional form of cross-sectional distribution
- **Cons: simulation**
  - Cross-sectional moments calculated inefficiently (Monte-Carlo integration)
  - Points at which aggregate law is fitted selected inefficiently. Recall that the standard error is equal to $\sigma^2(X'X)^{-1}$

B: Algorithms with parameterized cross-sectional distribution (Den Haan (1997), Reiter (2003))

- Evaluates the aggregate law of motion on a grid of Chebyshev nodes (ensures uniform convergence of polynomial approximations)

- Uses quadrature procedures to calculate next period's moments

$\Longrightarrow$ **What do we need to do this?**

- Need to assume a functional form of cross-sectional distribution

- This is unknown $\Longrightarrow$ parameterize with flexible functional form with N_M parameters

B: Algorithms with parameterized cross-sectional distribution (Den Haan (1997), Reiter (2003))

Den Haan (1997): distribution approximated with flexible functional form with N_M parameters. N_M moments used to pin down parameters

- **Disavantage**
  - High $N_M \Longrightarrow$ many state variables
  - Low $N_M \Longrightarrow$ inaccurate shape for cross-sectional distribution

This Algorithm

$\Longrightarrow$ **Use Reiter (2003) to improve projections algorithm**

- Use N_M moments as state variables (Reduction of the state space)
- Uses simulation to get information on the $N_{\overline{M}} - N_M + 1$ higher-order moments to get the shape of the cross-sectional distribution right

Useful contributions for other applications

1. we develop a simulation procedure that avoids cross-sectional sampling variation
2. we propose a particular class of parameterizing densities that makes the problem of finding the coefficients that correspond to a set of moments a convex optimization problem.
3. we provide a set of accuracy tests (alternatives of the R2, see Den Haan 2007)

Algorithm: General Overview

$\Longrightarrow$ Define a set of moments for which you calculate the transition law

$$m = \left[m^{u,c}, m^{e,1}, m^{u,1} \right]$$

$\Longrightarrow$ Iterative procedure

1. Calculate individual policies given the aggregate law of motion

$$m^{e,1'} = \Gamma^{e}(m, a, a'),\ m^{u,1'} = \Gamma^{u}(m, a, a'),\ m^{u,c'} = \Gamma^{u,c}(m, a, a')$$

2. Given solutions for individual policy rules, up-date aggregate laws

Algorithm details I

Procedure to solve for the aggregate laws of motion

1. Choose a grid of the aggregate state variable (Chebyshev nodes): "x values"

2. Using quadrature methods, calculate end-of-period moments, $\widetilde{m}^{w,j}$ for $j \in \{c, 1\}$ at each grid point and then we deduce $m^{w,j}$: "y values"

$$\widetilde{m}^{e,1} = (1 - m^{e,c}) \int k^{e}(k,s) P(k,\rho^{e}) dk + m^{e,c}.k^{e}(0,s)$$

3. Perform a projection step to find the coefficients of $\Gamma^{w}(m,a,a')$

Algorithm details I

Procedure to solve for the aggregate laws of motion

$\Longrightarrow$ Key issue at this stage:

Define the approximating densities (for positive asset holdings) and the number of moments characterizing these densities

- Exponential of polynomials $P(k, \rho^e)$ and $P(k, \rho^u)$
- Order $N_{\overline{M}}$ with $N_{\overline{M}} > N_M$

$\Longrightarrow$ Simulation techniques to get information on higher-order moments and define accurately the functional form

Algorithm details II

Simulation overview: generate reference moments

Get rid of sampling variation !

Given:

- Individual policy functions $g(k^i, a^i, a)$
- Initial cross-sectional distribution for *continuum* of agents
- Stochastic process for ε
- A time series of aggregate productivity shocks, $\{a_t\}_{t=1}^T$

Algorithm details II

Simulation overview: generate reference moments

1. Calculate the first $N_{\overline{M}}$ of next period's moments
2. Fit an $N_{\overline{M}}$ th-order polynomial to approximate cross-sectional distribution

Algorithm details II

Simulation overview: generate reference moments

$$\int_0^\infty P(k;\rho^w)dk = 1$$

$$\int_0^\infty k\,P(k;\rho^w)dk = m^{w,1}$$

$$\int_0^\infty \left[(k - m^{w,1})^j\right]\,P(k;\rho^w)dk = m^{w,j},\ j = 2, ..., N_{\overline{M}}$$

Strength is in one detail: Good functional form

$$P(k,\rho^{w}) = \rho_0^w \exp\left(\begin{array}{c} \rho_1^w\left[k-m^{w,1}\right]+ \\ \rho_2^w\left[\left(k-m^{w,1}\right)^2-m^{w,2}\right]+\cdots+ \\ \rho_{N_{\overline{M}}}^w\left[\left(k-m^{w,1}\right)^{N_{\overline{M}}}-m^{w,N_{\overline{M}}}\right]\end{array}\right).$$

Algorithm details II

Simulation overview: generate reference moments

Coefficients are solution to convex optimization problem

$$\min_{\rho_1^w,\rho_2^w,\cdots,\rho_{N_{\overline{M}}}^w} \int_0^\infty P(k,\rho^w)dk.$$

Accuracy of the results

Calibration and numerical details

- Krusell and Smith (1998) benchmark economy

$$u^g = 4\%, u^b = 10\%, a^g = 1.01, a^b = 0.99$$

- Cross-sectional distribution defined by 6 moments

Here we only focus on the accuracy of the simulation procedure

Accuracy of the results I

Comparison between MC simulation and the new simulation procedure

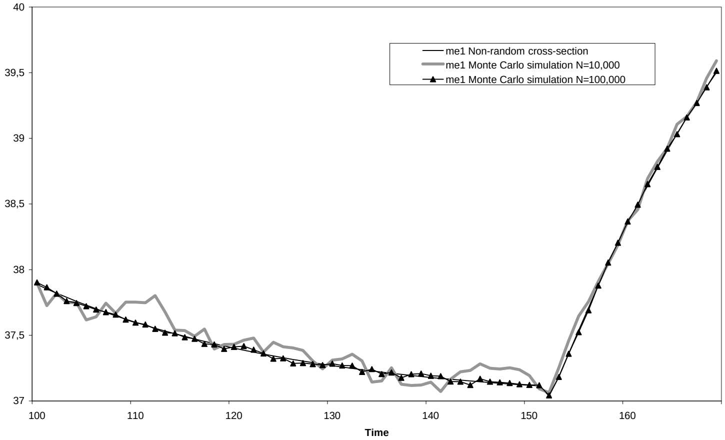

Figure: $m^{e,1}$ generated using a finite and a continuum of agents when the economy goes from bad to good state

Accuracy of the results II

Comparison between MC simulation and the new simulation procedure

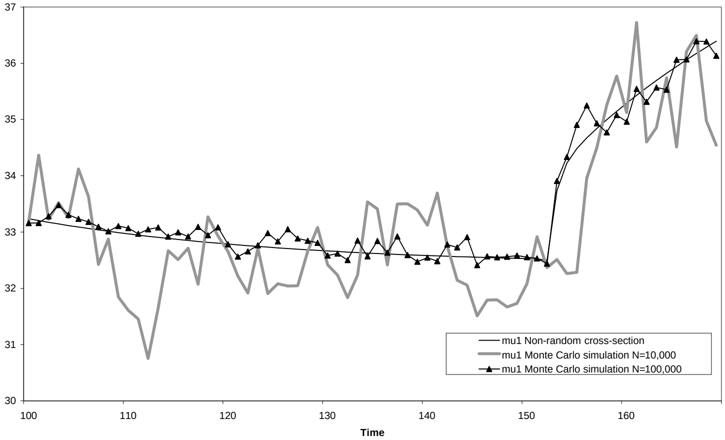

Figure: $m^{u,1}$ generated using a finite and a continuum of agents when the economy goes from bad to good state

Accuracy of the results III

Comparison between MC simulation and the new simulation procedure

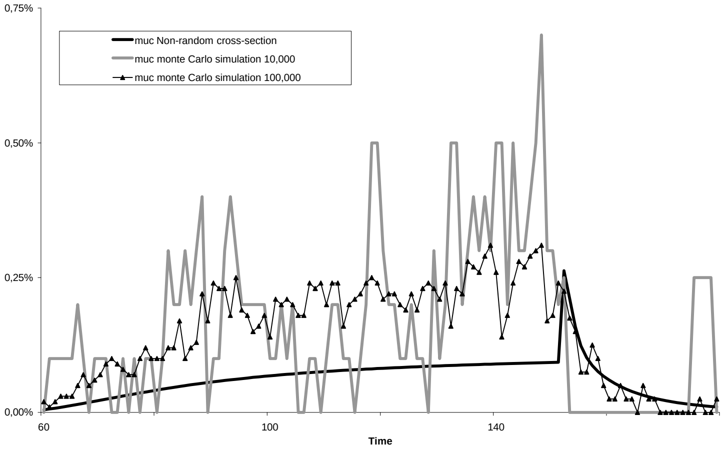

Figure: $m^{u,c}$ generated using a finite and a continuul of agents

Accuracy of the results IV

Accuracy of the densities: increasing the number of reference moments

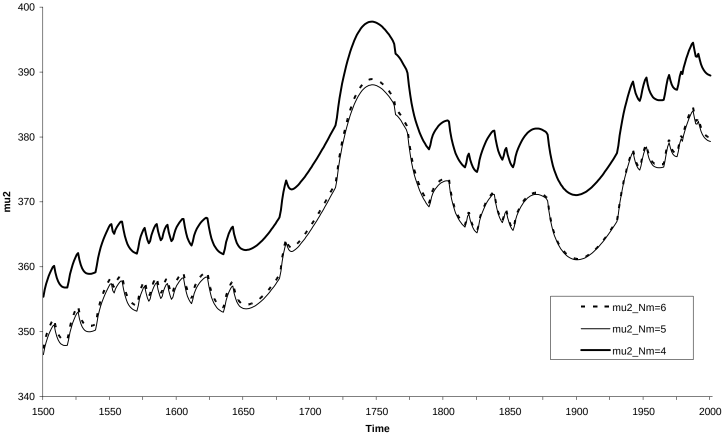

Figure: $m^{u,2}$ generated using a continuum of agents with different values of $N_{\overline{M}}$

Accuracy of the results IV

Shape of the distribution

Table: Differences between implied and actual higher-order moments using sixth-order approximating density

$N_{\overline{M}} = 6$	Employed		Unemployed	
Error (%)	Average	Max	Average	Max
$\frac{\left\lvert \overleftarrow{\Im}^{w,7} - \overleftarrow{m}^{w,7} \right\rvert}{\overleftarrow{m}^{w,7}}$	2.8E-2%	7.3E-1%	1.0E-1%	2.2E-1%
$\frac{\left\lvert \overleftarrow{\Im}^{w,8} - \overleftarrow{m}^{w,8} \right\rvert}{\overleftarrow{m}^{w,8}}$	4.3E-2%	1.0E-1%	1.8E-1%	4.3E-1%
$\frac{\left\lvert \overleftarrow{\Im}^{w,9} - \overleftarrow{m}^{w,9} \right\rvert}{\overleftarrow{m}^{w,9}}$	9.3E-2%	2.3E-1%	3.8E-1%	8.8E-1%
$\frac{\left\lvert \overleftarrow{\Im}^{w,10} - \overleftarrow{m}^{w,10} \right\rvert}{\overleftarrow{m}^{w,10}}$	1.3E-1%	3.1E-1%	5.6E-1%	1.3%

Accuracy of the transition laws

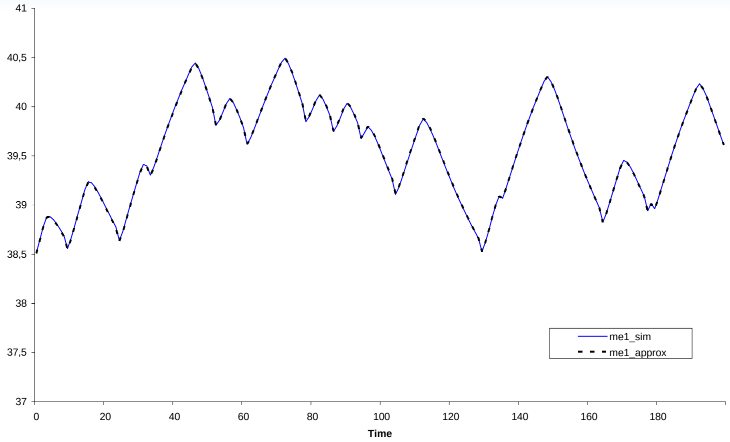

Figure: $m^{e,1}$ generated using the approximation $\Gamma^e(s)$ or the simulation on a continuum of agents

Conclusion

- Improvement upon traditional projection techniques
- New simulation techniques and approximating densities which could also be worthwhile if you use simulation to calculate the transition laws of the moments.
  - Economies where the unemployed become entrepreneurs
  - Policy evaluation: need to be really accurate to gauge the persistence of policy shocks

Example

Algan et al. (2007): Monetary shocks with incomplete markets and heterogeneous agents

- Monetary shocks in a Bewley style model where money is the only asset used for self-insurance
- Non-neutrality and persistence of monetary shocks only due to incomplete markets: alternative to sticky prices

Example

Algan et al. (2007): Monetary shocks with incomplete markets and heterogeneous agents

The recursive program of the household expressed by in real terms is

$$v\left(m_{t-1}, s_t; \gamma_t, \bar{M}_{t-1}\right) = \max_{m_t, c_t} \; u\left(c_t, 1 - l_t\right) + \beta E_t \left[v\left(m_t, s_{t+1}; \gamma_{t+1}, \bar{M}_t\right) | s_t,\right.$$

subject to the budget constraints

$$\begin{aligned} c_t + m_t &= \frac{m_{t-1}}{\Pi_t} + w_t l_t \varepsilon_t + b_t (1 - \varepsilon_t) + \gamma_t \frac{\bar{M}_{t-1}}{\Pi_t} \\ m_t &\geq 0 \end{aligned}$$

and

$$\ln\left(\bar{M}_t\right) = a_0^i + a_1^i \ln\left(\bar{M}_{t-1}\right)$$

Tricky thing here:

- Need to iterate at each period on the inflation rate to find the equilibrium inflation rate
- Get rid of sampling variation to gauge the persistence of monetary shocks

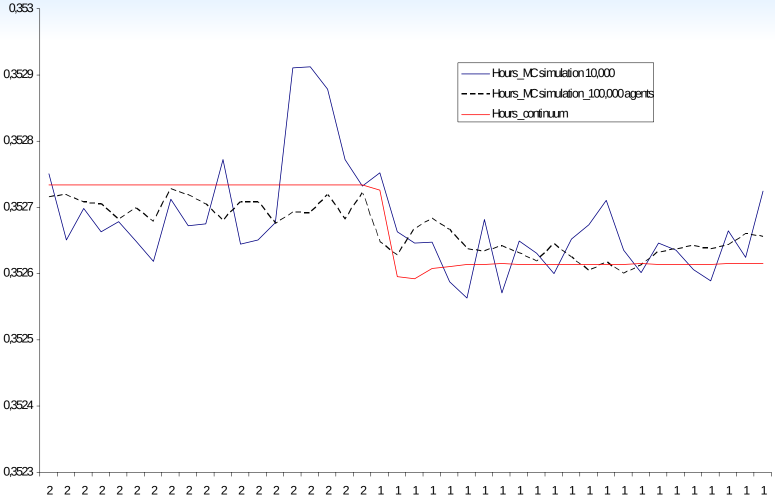

Figure: Impulse response of hours under different simulation procedures

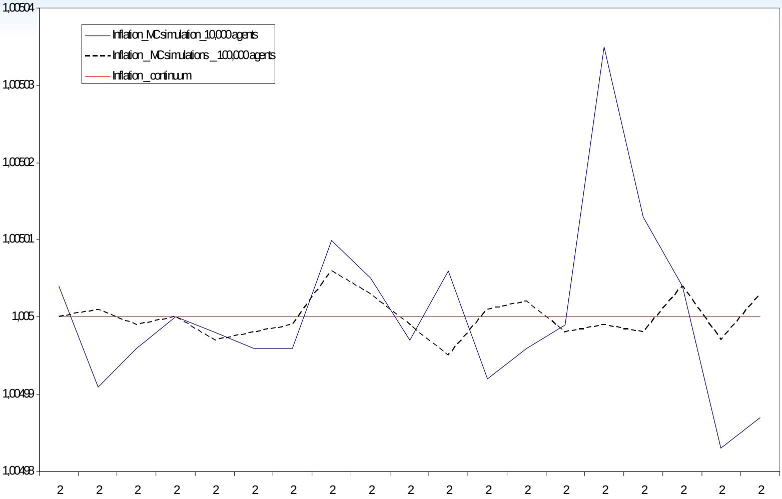

Figure: Inflation Rate

Summary

- Algorithm uses classic elements of numericals solutions literature but rectangular grid is problematic
- Perturbation techniques may be the way to go (Reiter (2006) and Preston and Roca(2007))
- How to test accuracy: Den Haan (2007)

How to access accuracy?

- Standard procedure: R-square
- Problems:
  - In sample fit ("truth" is used to generate explanatory variable mt)
  - An average (may hide large errors)
  - Scales errors by variance of dependent variable

Den Haan (2007)

- Truth: $m_{t+1} = \alpha_0 + \alpha_{1mt} + \alpha_2 a_t + \alpha_3 m_{t-1}$
- Approximation: $m_{t+1} = \gamma_0 + \gamma_{1mt} + \gamma_2 a_t$

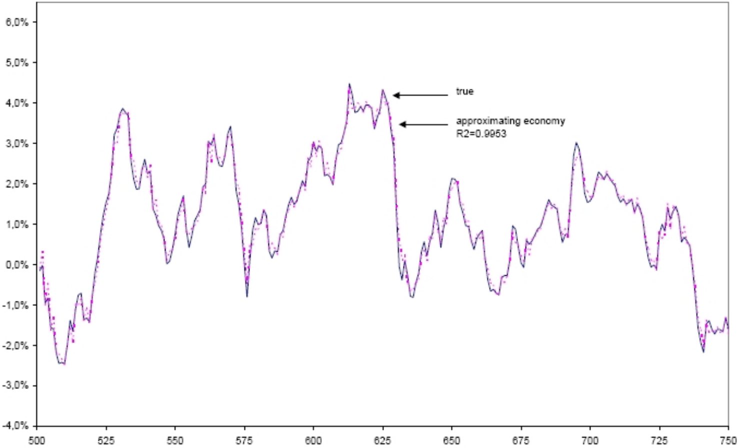

Figure: In sample fit

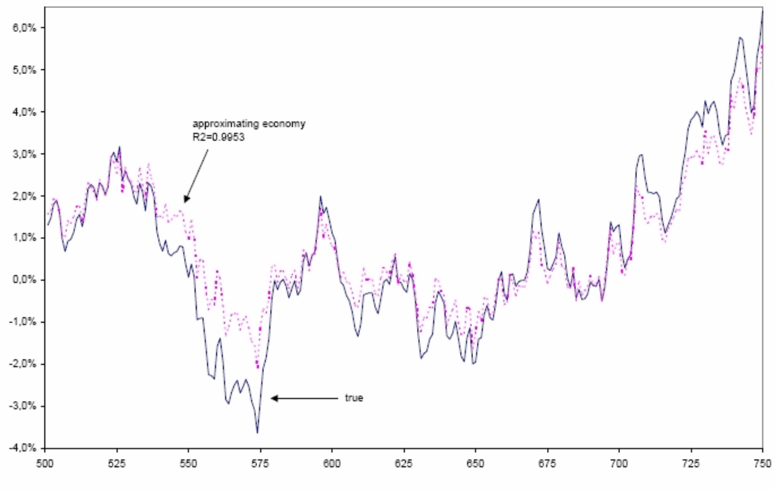

Figure: Independently generated

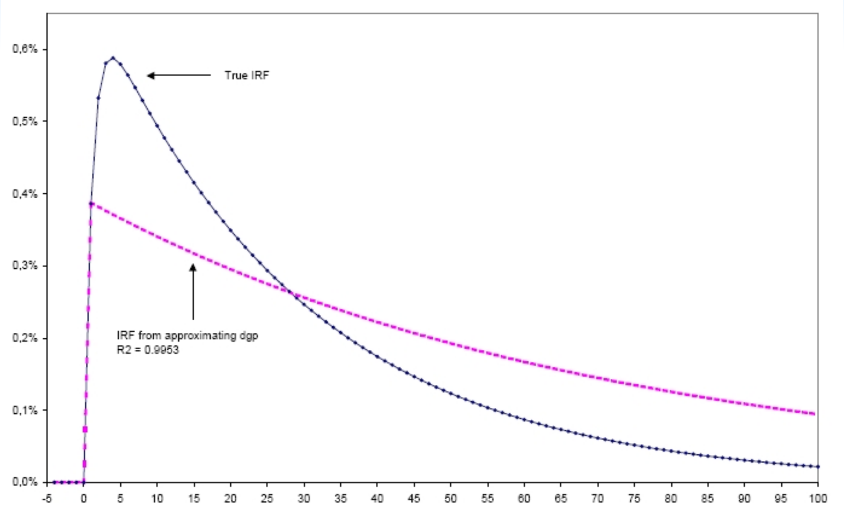

Figure: Impulse Response Functions