Computing Stochastic Dynamic Economic Models with a Large Number of State Variables: A Description and Application of a Smolyak-Collocation Method

Dirk Krueger – University of Pennsylvania, CEPR and NBER
Felix Kubler – University of Pennsylvania
Ben Malin – Federal Reserve Board

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Economic Model

- State variables: $s \in \mathcal{S} \subset \Re^d$
- Equilibrium: Policy functions $F(s)$ satisfy

$$g\left(F(s), s\right) = \mathbf{E}_{\tilde{s}}\left[h\left(F(\tilde{s}), \tilde{s}\right)\right]$$

  for all $s \in \mathcal{S}$, where $\tilde{s} = i(F(s), s)$.

- Want an algorithm to approximate F when $\mathcal{S}$ is high-dimensional.

A Solution Method - Time Iteration Collocation

1: Select a d-dimensional box $\mathcal{S}$ and a family of functions $\hat{F}$.

2: Select a finite grid $\mathcal{H} \subset \mathcal{S}$ of collocation points and a starting $\hat{F}^0$.

3: Given the function $\hat{F}^n$, $\forall s \in \mathcal{H}$, solve system

$$g(x, s) = \mathbf{E}_{\tilde{s}} \left[h \left(\hat{F}^n(\tilde{s}), \tilde{s} \right) \right]$$

for the unknown x.

4: Compute the new function $\hat{F}^{n+1}$ by interpolation of the solutions in 3.

5: Check some stopping criterion; if not satisfied, go to 3.

What is Needed?

- High-Dimensional Interpolation
  * Grid of Collocation Points: $\mathcal{H}$
  * Family of Approximating Functions: $\hat{F}$
  * Interpolation Formula for Computing Coefficients of $\hat{F}$

What is Needed?

- High-Dimensional Interpolation
  - Grid of Collocation Points: $\mathcal{H}$
  - Family of Approximating Functions: $\hat{F}$
  - Interpolation Formula for Computing Coefficients of $\hat{F}$
- Bounds for State Space: $\mathcal{S}$
- Initial Guess: $\hat{F}^0$
- Method for Solving System of Nonlinear Equations
- High-Dimensional Integration - Use Monomial Formula

Smolyak-Collocation Method: Overview

- *Global* Approximation
- Innovation to Economics Literature: Sparse Grid
  - Number of grid points grows polynomially in d.
  - Allows for projection methods for $d > 3$.

Smolyak's Algorithm - Collocation Points

Barthelmann, Novak and Ritter (Advances of Computational Mathematics 12, 1999):

- Assume we want to approximate $f : [-1,1]^d \to \Re$.
- Define $m_1 = 1$ and $m_i = 2^{i-1} + 1$, $i = 2, ...$
- Define $\mathcal{G}^i = \{\zeta_1^i, ..., \zeta_{m_i}^i\} \subset [-1,1]$ as the set of the extrema of the Chebyshev polynomials

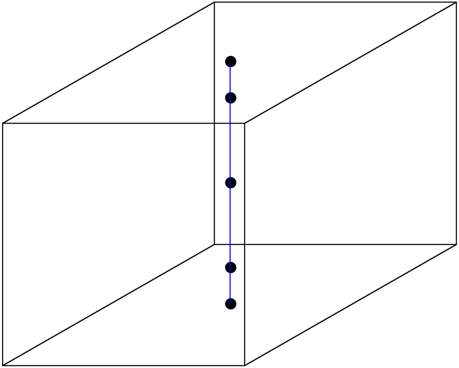
$$\zeta_j^i = -cos\left(\frac{\pi(j-1)}{m_i - 1}\right) \quad j = 1, ..., m_i$$

  with $\mathcal{G}^1 = \{0\}$. It is crucial that $\mathcal{G}^i \subset \mathcal{G}^{i+1}$, $\forall i = 1, 2, \ldots$
- For d and approximation level λ, define a sparse grid

$$\mathcal{H}^{d,\lambda} = \bigcup_{\mathbf{i}:|\mathbf{i}|=d+\lambda} (\mathcal{G}^{i_1} \times ... \times \mathcal{G}^{i_d}),$$

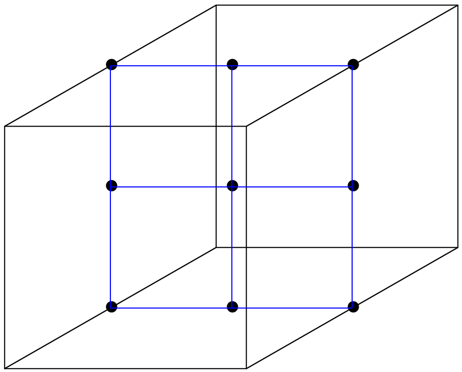
  where $|\mathbf{i}| = i_1 + \ldots + i_d$.

Collocation Points: $\mathcal{H}^{3,2}$



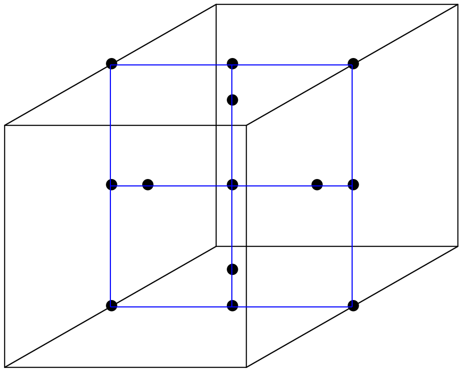
$$\mathcal{G}^3 \times \mathcal{G}^1 \times \mathcal{G}^1$$

Collocation Points: $\mathcal{H}^{3,2}$



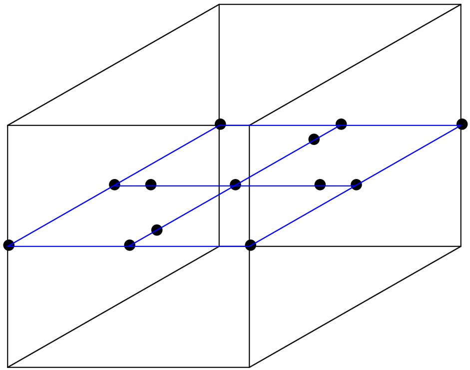
$$\mathcal{G}^2 \times \mathcal{G}^2 \times \mathcal{G}^1$$

Collocation Points: $\mathcal{H}^{3,2}$



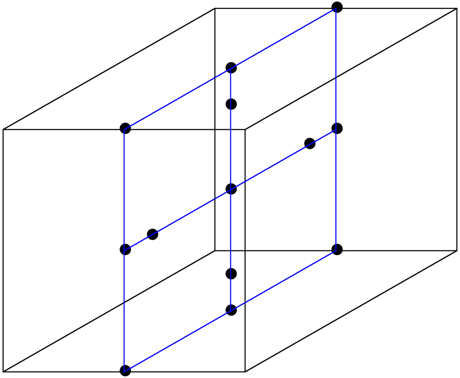
$$\mathcal{G}^3 \times \mathcal{G}^1 \times \mathcal{G}^1 \bigcup \mathcal{G}^1 \times \mathcal{G}^3 \times \mathcal{G}^1 \bigcup \mathcal{G}^2 \times \mathcal{G}^2 \times \mathcal{G}^1$$

Collocation Points: $\mathcal{H}^{3,2}$



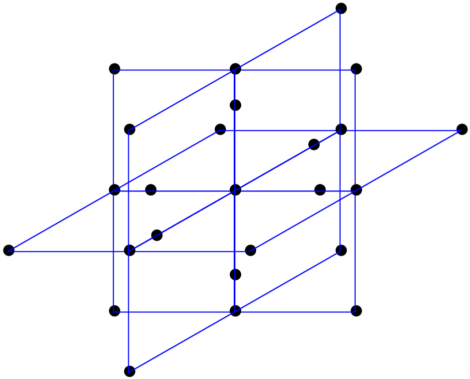
$$\mathcal{G}^1 \times \mathcal{G}^3 \times \mathcal{G}^1 \bigcup \mathcal{G}^1 \times \mathcal{G}^1 \times \mathcal{G}^3 \bigcup \mathcal{G}^1 \times \mathcal{G}^2 \times \mathcal{G}^2$$

Collocation Points: $\mathcal{H}^{3,2}$



$$\mathcal{G}^3 \times \mathcal{G}^1 \times \mathcal{G}^1 \bigcup \mathcal{G}^1 \times \mathcal{G}^1 \times \mathcal{G}^3 \bigcup \mathcal{G}^2 \times \mathcal{G}^1 \times \mathcal{G}^2$$

Collocation Points: $\mathcal{H}^{3,2}$



Number of Points: $1 + 4d + 4\dfrac{d(d-1)}{2}$

Smolyak's Algorithm - Interpolation

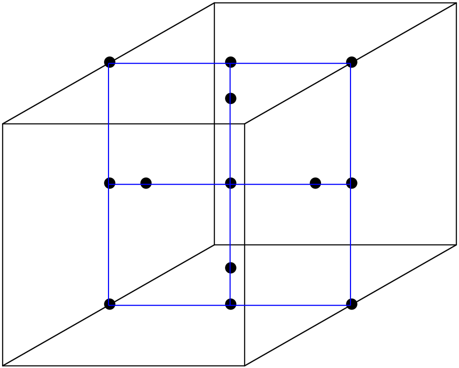
- Define $p^{\mathbf{i}}$ as the tensor-product multivariate polynomial which interpolates on $\mathcal{G}^{m(i_1)} \times \ldots \times \mathcal{G}^{m(i_d)}$.

$$p^{\mathbf{i}}(x) = \sum_{l_1=1}^{m(i_1)} \cdots \sum_{l_d=1}^{m(i_d)} \theta_{l_1 \ldots l_d}(f) T_{l_1}(x_1) \cdots T_{l_d}(x_d)$$

- The approximating function is given by

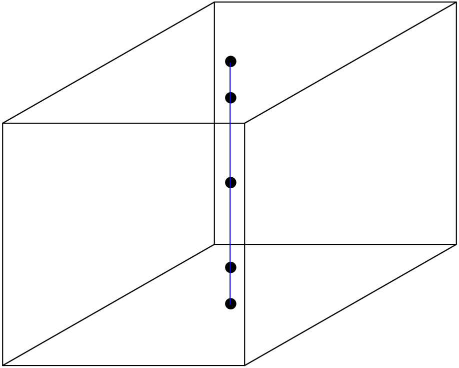
$$\hat{f}^{d,\lambda}(x) = \sum_{max(d,\lambda+1) \leq |\mathbf{i}| \leq d+\lambda} (-1)^{d+\lambda-|\mathbf{i}|} \begin{pmatrix} d-1 \\ d+\lambda-|\mathbf{i}| \end{pmatrix} p^{\mathbf{i}}(x)$$

Weighting Polynomials



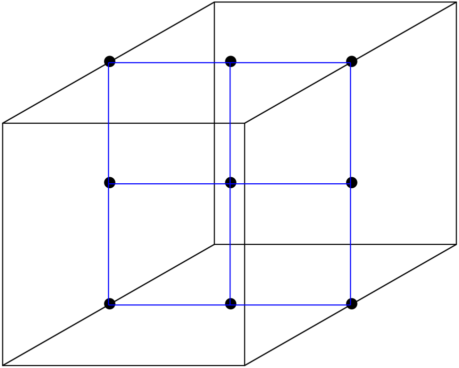
$$\mathcal{G}^3 \times \mathcal{G}^1 \times \mathcal{G}^1 \bigcup \mathcal{G}^1 \times \mathcal{G}^3 \times \mathcal{G}^1 \bigcup \mathcal{G}^2 \times \mathcal{G}^2 \times \mathcal{G}^1$$

Weighting Polynomials: x_1^4



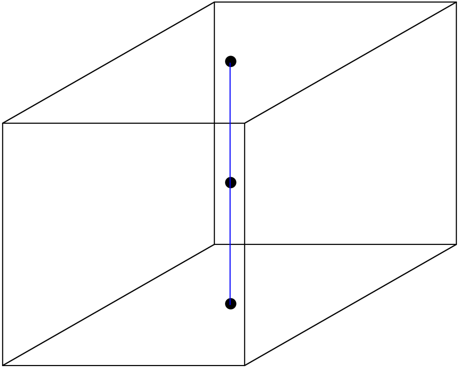
$$(-1)^{d+\lambda-|\mathbf{i}|}\binom{d-1}{d+\lambda-|\mathbf{i}|}=1$$

Weighting Polynomials: $x_1^2 x_2^2$



$$(-1)^{d+\lambda-|\mathbf{i}|} \binom{d-1}{d+\lambda-|\mathbf{i}|} = 1$$

Weighting Polynomials: x_1^2



$$(-1)^{d+\lambda-|\mathbf{i}|}\binom{d-1}{d+\lambda-|\mathbf{i}|} = -2$$

Model Specifications

Mod	N	Volatility	ϕ	γ	η	χ	μ
A1	2,6	L, H	.5,10	1	–	–	–
A2	2,6	L, H	.5,10	.25, 1	.1, 1	–	–
A3	2,6	L, H	.5,10	.25, 1	–	–	–
A4	2,6	L, H	.5,10	.25	–	.83	-0.2
A5	2,6	L, H	.5,10	(.25,1)	–	–	–
A6	2,6	L, H	.5,10	(.25,1)	(.1,1)	–	–
A7	2,4	L, H	.5,10	(.25,1)	–	–	–
A8	2,4	L, H	.5,10	(.2,.4)	–	(.75,.9)	(-.3,.3)

Accuracy Tests: $N = 2$

	Test 1			Test 2		Test 3	
	$T_{.01}$	$T_{.1}$	$T_{.3}$	S_{max}	S_{mean}	$P_{.05}$	$P_{.95}$
A1	-6.0	-5.1	-4.2	-5.2	-5.8	.04	.06
A2	-5.4	-4.7	-4.1	-4.7	-5.3	.03	.09
A3	-5.3	-4.3	-3.7	-4.2	-5.0	.03	.07
A4	-5.2	-4.3	-3.7	-4.3	-4.9	.03	.07
A5	-5.8	-4.9	-4.0	-5.1	-5.6	.04	.06
A6	-5.8	-4.7	-3.9	-4.9	-5.6	.04	.06
A7	-5.3	-4.3	-3.6	-4.2	-4.8	.02	.07
A8	-4.9	-4.2	-3.6	-4.1	-4.6	.03	.07

Accuracy Tests: $N = 6$

	Test 1			Test 2		Test 3	
	$T_{.01}$	$T_{.1}$	$T_{.3}$	S_{max}	S_{mean}	$P_{.05}$	$P_{.95}$
A1	-5.9	-5.2	-4.6	-5.1	-5.8	0	.47
A2	-5.3	-4.8	-4.5	-4.7	-5.3	0	.51
A3	-5.4	-4.6	-4.0	-4.1	-4.8	0	.55
A4	-5.3	-4.2	-3.1	-3.4	-4.0	0	.55
A5	-5.5	-5.1	-4.6	-5.0	-5.6	0	.48
A6	-4.4	-4.3	-3.9	-4.4	-4.5	0	.53
A7	–	–	–	–	–	–	–
A8	-4.9	-4.1	-3.7	-4.2	-4.7	0	.39

N = 4 for A7-A8

Running Times: Selected Specifications

	N	Time(seconds)			
		Sol.	**Test 1**	**Test 2**	**Test 3**
A2	2	1.5	0.02	0.08	2.2
A4	2	5.1	0.8	1.2	16.9
A8	2	70	0.8	1.2	16.9
A2	6	1676	1.7	2.6	977
A4	6	3297	19.9	28.9	1039
A8	4	2603	5.7	8.5	70

Extra Slides

Accuracy Tests - Curvature

	γ	η	Test 1 $\mathbf{T}_{.01}$	$\mathbf{T}_{.1}$	$\mathbf{T}_{.3}$	Test 2 $\mathbf{S}_{max}$	$\mathbf{S}_{mean}$	Test 3 $\mathbf{P}_{.05}$	$\mathbf{P}_{.95}$
N = 2									
A2	0.25	0.1	-5.4	-4.7	-4.1	-4.7	-5.3	.03	.09
A2	1.0	1.0	-6.0	-4.8	-3.7	-4.6	-5.5	.03	.08
A3	0.25	–	-5.3	-4.3	-3.7	-4.2	-5.0	.03	.07
A3	1.0	–	-5.8	-4.6	-3.7	-4.4	-5.3	.03	.06
N = 6									
A2	0.25	0.1	-5.3	-4.8	-4.5	-4.7	-5.3	0	.51
A2	1.0	1.0	-5.9	-4.9	-4.0	-4.4	-5.3	0	.58
A3	0.25	–	-5.4	-4.6	-4.0	-4.1	-4.8	0	.55
A3	1.0	–	-5.8	-4.8	-4.0	-4.3	-5.1	0	.62

Accuracy Tests: $N = 2$, $\phi = 10$

	Test 1			Test 2		Test 3	
	$T_{.01}$	$T_{.1}$	$T_{.3}$	S_{max}	S_{mean}	$P_{.05}$	$P_{.95}$
A1	-5.6	-5.2	-4.5	-5.3	-5.7	.02	.14
A2	-5.6	-5.1	-4.5	-5.2	-5.7	.02	.15
A3	-5.5	-4.8	-4.3	-4.8	-5.4	.01	.15
A4	-5.4	-4.5	-3.7	-4.6	-5.2	.01	.14
A5	-5.7	-5.0	-4.3	-5.1	-5.6	.01	.13
A6	-5.6	-5.0	-4.3	-5.1	-5.6	.02	.14
A7	-5.1	-4.4	-4.1	-4.4	-4.9	.01	.16
A8	-4.9	-4.2	-3.8	-4.2	-4.6	.01	.15

Accuracy Tests: $N = 6$, $\phi = 10$

	Test 1			Test 2		Test 3	
	$T_{.01}$	$T_{.1}$	$T_{.3}$	S_{max}	S_{mean}	$P_{.05}$	$P_{.95}$
A1	-5.2	-4.9	-4.7	-5.0	-5.3	0	.99
A2	-5.3	-4.8	-4.5	-4.7	-5.3	0	.98
A3	-5.2	-4.7	-4.4	-4.7	-5.3	0	.99
A4	-5.2	-4.7	-4.3	-4.6	-5.2	0	.99
A5	-5.3	-5.0	-4.7	-5.0	-5.5	0	.99
A6	-4.8	-4.6	-4.5	-4.7	-4.9	0	.99
A7	–	–	–	–	–	–	–
A8	-5.0	-4.1	-3.9	-4.2	-4.7	0	.83

N = 4 for A7-A8

Policy Functions

- Capital fairly linear in all state variables.
- Consumption
  - Concave in k
  - Convex in $log(a)$
- Labor highly nonlinear w.r.t own capital.