

# Real Business Cycles

Jesús Fernández-Villaverde  
University of Pennsylvania

## Household Problem

- Preferences:

$$\max E \sum_{t=0}^{\infty} \beta \{ \log c_t + \psi \log (1 - l_t) \}$$

- Budget constraint:

$$c_t + k_{t+1} = w_t l_t + r_t k_t + (1 - \delta) k_t, \forall t > 0$$

## Problem of the Firm

- Neoclassical production function:

$$y_t = k_t^\alpha (e^{z_t} l_t)^{1-\alpha}$$

- By profit maximization:

$$\begin{aligned} \alpha k_t^{\alpha-1} (e^{z_t} l_t)^{1-\alpha} &= r_t \\ (1 - \alpha) k_t^\alpha (e^{z_t} l_t)^{1-\alpha} l_t^{-1} &= w_t \end{aligned}$$

## Evolution of the technology

- $z_t$  changes over time.
- It follows the AR(1) process:

$$z_t = \rho z_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim \mathcal{N}(0, \sigma)$$

- Interpretation of  $\rho$ .

## A Competitive Equilibrium

- We can define a competitive equilibrium in the standard way.
- The competitive equilibrium is unique.
- This economy satisfies the conditions that assure that both welfare theorems hold.
- Why is this important? We could solve instead the Social Planner's Problem associated with it.
- Advantages and disadvantages of solving the social planner's problem.

## The Social Planner's Problem

- It has the form:

$$\max E \sum_{t=0}^{\infty} \beta \{ \log c_t + \psi \log (1 - l_t) \}$$

$$c_t + k_{t+1} = k_t^\alpha (e^{z_t} l_t)^{1-\alpha} + (1 - \delta) k_t, \forall t > 0$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \varepsilon_t \sim \mathcal{N}(0, \sigma)$$

- This is a dynamic optimization problem.

## Computing the RBC

- The previous problem does not have a known “paper and pencil” solution.
- We will work with an approximation: Perturbation Theory.
- We will undertake a first order perturbation of the model.
- How well will the approximation work?

## Equilibrium Conditions

From the household problem+firms's problem+aggregate conditions:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left( 1 + \alpha k_t^{\alpha-1} (e^{z_t} l_t)^{1-\alpha} - \delta \right) \right\}$$

$$\psi \frac{c_t}{1-l_t} = (1-\alpha) k_t^\alpha (e^{z_t} l_t)^{1-\alpha} l_t^{-1}$$

$$c_t + k_{t+1} = k_t^\alpha (e^{z_t} l_t)^{1-\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$



## Finding a Deterministic Solution

- We search for the first component of the solution.
- If  $\sigma = 0$ , the equilibrium conditions are:

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} \left( 1 + \alpha k_t^{\alpha-1} l_t^{1-\alpha} - \delta \right)$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} l_t^{-\alpha}$$
$$c_t + k_{t+1} = k_t^{\alpha} l_t^{1-\alpha} + (1 - \delta) k_t$$

## Deterministic Steady State

- The equilibrium conditions imply a steady state:

$$\begin{aligned}\frac{1}{c} &= \beta \frac{1}{c} (1 + \alpha k^{\alpha-1} l^{1-\alpha} - \delta) \\ \psi \frac{c}{1-l} &= (1 - \alpha) k^{\alpha} l^{-\alpha} \\ c + \delta k &= k^{\alpha} l^{1-\alpha}\end{aligned}$$

- Or simplifying:

$$\begin{aligned}\frac{1}{\beta} &= 1 + \alpha k^{\alpha-1} l^{1-\alpha} - \delta \\ \psi \frac{c}{1-l} &= (1 - \alpha) k^{\alpha} l^{-\alpha} \\ c + \delta k &= k^{\alpha} l^{1-\alpha}\end{aligned}$$

## Solving the Steady State

Solution:

$$k = \frac{\mu}{\Omega + \varphi\mu}$$

$$l = \varphi k$$

$$c = \Omega k$$

$$y = k^\alpha l^{1-\alpha}$$

where  $\varphi = \left(\frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta\right)\right)^{\frac{1}{1-\alpha}}$ ,  $\Omega = \varphi^{1-\alpha} - \delta$  and  $\mu = \frac{1}{\psi} (1 - \alpha) \varphi^{-\alpha}$ .

## Linearization I

- Loglinearization or linearization?
- Advantages and disadvantages
- We can linearize and perform later a change of variables.

## Linearization II

We linearize:

$$\begin{aligned}\frac{1}{c_t} &= \beta E_t \left\{ \frac{1}{c_{t+1}} \left( 1 + \alpha k_t^{\alpha-1} (e^{z_t} l_t)^{1-\alpha} - \delta \right) \right\} \\ \psi \frac{c_t}{1-l_t} &= (1-\alpha) k_t^\alpha (e^{z_t} l_t)^{1-\alpha} l_t^{-1} \\ c_t + k_{t+1} &= k_t^\alpha (e^{z_t} l_t)^{1-\alpha} + (1-\delta) k_t \\ z_t &= \rho z_{t-1} + \varepsilon_t\end{aligned}$$

around  $l$ ,  $k$ , and  $c$  with a First-order Taylor Expansion.

## Linearization III

We get:

$$\begin{aligned}
 -\frac{1}{c}(c_t - c) &= E_t \left\{ \begin{aligned} &-\frac{1}{c}(c_{t+1} - c) + \alpha(1 - \alpha)\beta\frac{y}{k}z_{t+1} + \\ &\alpha(\alpha - 1)\beta\frac{y}{k^2}(k_{t+1} - k) + \alpha(1 - \alpha)\beta\frac{y}{kl}(l_{t+1} - l) \end{aligned} \right\} \\
 \frac{1}{c}(c_t - c) + \frac{1}{(1 - l)}(l_t - l) &= (1 - \alpha)z_t + \frac{\alpha}{k}(k_t - k) - \frac{\alpha}{l}(l_t - l) \\
 (c_t - c) + (k_{t+1} - k) &= \left\{ \begin{aligned} &y \left( (1 - \alpha)z_t + \frac{\alpha}{k}(k_t - k) + \frac{(1 - \alpha)}{l}(l_t - l) \right) \\ &+ (1 - \delta)(k_t - k) \end{aligned} \right\} \\
 z_t &= \rho z_{t-1} + \varepsilon_t
 \end{aligned}$$

## Rewriting the System I

Or:

$$\alpha_1 (c_t - c) = E_t \{ \alpha_1 (c_{t+1} - c) + \alpha_2 z_{t+1} + \alpha_3 (k_{t+1} - k) + \alpha_4 (l_{t+1} - l) \}$$

$$(c_t - c) = \alpha_5 z_t + \frac{\alpha}{k} c (k_t - k) + \alpha_6 (l_t - l)$$

$$(c_t - c) + (k_{t+1} - k) = \alpha_7 z_t + \alpha_8 (k_t - k) + \alpha_9 (l_t - l)$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

## Rewriting the System II

where

$$\begin{aligned}\alpha_1 &= -\frac{1}{c} & \alpha_2 &= \alpha(1-\alpha)\beta\frac{y}{k} \\ \alpha_3 &= \alpha(\alpha-1)\beta\frac{y}{k^2} & \alpha_4 &= \alpha(1-\alpha)\beta\frac{y}{kl} \\ \alpha_5 &= (1-\alpha)c & \alpha_6 &= -\left(\frac{\alpha}{l} + \frac{1}{(1-l)}\right)c \\ \alpha_7 &= (1-\alpha)y & \alpha_8 &= y\frac{\alpha}{k} + (1-\delta) \\ \alpha_9 &= y\frac{(1-\alpha)}{l} & y &= k^\alpha l^{1-\alpha}\end{aligned}$$



## Rewriting the System III

After some algebra the system is reduced to:

$$A(k_{t+1} - k) + B(k_t - k) + C(l_t - l) + Dz_t = 0$$

$$E_t(G(k_{t+1} - k) + H(k_t - k) + J(l_{t+1} - l) + K(l_t - l) + Lz_{t+1} + Mz_t) = 0$$

$$E_t z_{t+1} = \rho z_t$$

## Guess Policy Functions

We guess policy functions of the form  $(k_{t+1} - k) = P(k_t - k) + Qz_t$  and  $(l_t - l) = R(k_t - k) + Sz_t$ , plug them in and get:

$$\begin{aligned} & A(P(k_t - k) + Qz_t) + B(k_t - k) \\ & + C(R(k_t - k) + Sz_t) + Dz_t = 0 \end{aligned}$$

$$\begin{aligned} & G(P(k_t - k) + Qz_t) + H(k_t - k) + J(R(P(k_t - k) + Qz_t) + SNz_t) \\ & + K(R(k_t - k) + Sz_t) + (LN + M)z_t = 0 \end{aligned}$$

## Solving the System I

Since these equations need to hold for any value  $(k_{t+1} - k)$  or  $z_t$  we need to equate each coefficient to zero, on  $(k_t - k)$ :

$$AP + B + CR = 0$$

$$GP + H + JRP + KR = 0$$

and on  $z_t$ :

$$AQ + CS + D = 0$$

$$(G + JR)Q + JSN + KS + LN + M = 0$$

## Solving the System II

- We have a system of four equations on four unknowns.
- To solve it note that  $R = -\frac{1}{C}(AP + B) = -\frac{1}{C}AP - \frac{1}{C}B$
- Then:

$$P^2 + \left( \frac{B}{A} + \frac{K}{J} - \frac{GC}{JA} \right) P + \frac{KB - HC}{JA} = 0$$

a quadratic equation on  $P$ .

## Solving the System III

- We have two solutions:

$$P = -\frac{1}{2} \left( -\frac{B}{A} - \frac{K}{J} + \frac{GC}{JA} \pm \left( \left( \frac{B}{A} + \frac{K}{J} - \frac{GC}{JA} \right)^2 - 4 \frac{KB - HC}{JA} \right)^{0.5} \right)$$

one stable and another unstable.

- If we pick the stable root and find  $R = -\frac{1}{C} (AP + B)$  we have to a system of two linear equations on two unknowns with solution:

$$Q = \frac{-D(JN + K) + CLN + CM}{AJN + AK - CG - CJR}$$

$$S = \frac{-ALN - AM + DG + DJR}{AJN + AK - CG - CJR}$$

## Calibration

- What does it mean to calibrate a model?
- Our choices

### Calibrated Parameters

Parameter	$\beta$	$\psi$	$\alpha$	$\delta$	$\rho$	$\sigma$
Value	0.99	1.75	0.33	0.023	0.95	0.01