

# Implementation of Ramsey Optimal Policy in Dynare++, Timeless Perspective

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**Abstract:** This document provides a derivation of Ramsey optimal policy from timeless perspective and describes its implementation in Dynare++.

## 1 Derivation of the First Order Conditions

Let us start with an economy populated by agents who take a number of variables exogenously, or given. These may include taxes or interest rates for example. These variables can be understood as decision (or control) variables of the timeless Ramsey policy (or social planner). The agent's information set at time  $t$  includes mass-point distributions of these variables for all times after  $t$ . If  $i_t$  denotes an interest rate for example, then the information set  $I_t$  includes  $i_{t|t}, i_{t+1|t}, \dots, i_{t+k|t}, \dots$  as numbers. In addition the information set includes all realizations of past exogenous innovations  $u_\tau$  for  $\tau = t, t-1, \dots$  and distributions  $u_\tau \sim N(0, \Sigma)$  for  $\tau = t+1, \dots$ . These information sets will be denoted  $I_t$ .

An information set including only the information on past realizations of  $u_\tau$  and future distributions of  $u_\tau \sim N(0, \sigma)$  will be denoted  $J_t$ . We will use the following notation for expectations through these sets:

$$\begin{aligned} E_t^I[X] &= E(X|I_t) \\ E_t^J[X] &= E(X|J_t) \end{aligned}$$

The agents optimize taking the decision variables of the social planner at  $t$  and future as given. This means that all expectations they form are conditioned on the set  $I_t$ . Let  $y_t$  denote a vector of all endogenous variables including the planner's decision variables. Let the number of endogenous variables be  $n$ . The economy can be described by  $m$  equations including the first order conditions and transition equations:

$$E_t^I [f(y_{t-1}, y_t, y_{t+1}, u_t)] = 0. \quad (1)$$

This leaves  $n - m$  the planner's control variables. The solution of this problem is a decision rule of the form:

$$y_t = g(y_{t-1}, u_t, c_{t|t}, c_{t+1|t}, \dots, c_{t+k|t}, \dots), \quad (2)$$

where  $c$  is a vector of planner's control variables.

Each period the social planner chooses the vector  $c_t$  to maximize his objective such that (2) holds for all times following  $t$ . This would lead to  $n - m$  first order conditions with respect to  $c_t$ . These first order conditions would contain unknown derivatives of endogenous variables with respect to  $c$ , which would have to be retrieved from the implicit constraints (1) since the explicit form (2) is not known.

The other way to proceed is to assume that the planner is so dumb that he is not sure what are his control variables. So he optimizes with respect to all  $y_t$  given the constraints (1). If the planner's objective is  $b(y_{t-1}, y_t, y_{t+1}, u_t)$  with a discount rate  $\beta$ , then the optimization problem looks as follows:

$$\begin{aligned} \max_{\{y_\tau\}_t^\infty} E_t^J \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} b(y_{\tau-1}, y_\tau, y_{\tau+1}, u_\tau) \right] \\ \text{s.t.} \quad E_\tau^I [f(y_{\tau-1}, y_\tau, y_{\tau+1}, u_\tau)] = 0 \quad \text{for } \tau = \dots, t-1, t, t+1, \dots \end{aligned} \quad (3)$$

Note two things: First, each constraint (1) in (3) is conditioned on  $I_\tau$  not  $I_t$ . This is very important, since the behaviour of agents at period  $\tau = t + k$  is governed by the constraint using expectations conditioned on  $t + k$ , not  $t$ . The social planner knows that at  $t + k$  the agents will use all information available at  $t + k$ . Second, the constraints for the planner's decision made at  $t$  include also constraints for agent's behaviour prior to  $t$ . This is because the agent's decision rules are given in the implicit form (1) and not in the explicit form (2).

Using Lagrange multipliers, this can be rewritten as

$$\begin{aligned} \max_{y_t} E_t^J \left[ \sum_{\tau=t}^{\infty} \beta^{\tau-t} b(y_{\tau-1}, y_\tau, y_{\tau+1}, u_\tau) \right. \\ \left. + \sum_{\tau=-\infty}^{\infty} \beta^{\tau-t} \lambda_\tau^T E_\tau^I [f(y_{\tau-1}, y_\tau, y_{\tau+1}, u_\tau)] \right], \end{aligned} \quad (4)$$

where  $\lambda_t$  is a vector of Lagrange multipliers corresponding to constraints (1). Note that the multipliers are multiplied by powers of  $\beta$  in order to make them stationary. Taking a derivative wrt  $y_t$  and putting it to zero yields the first order conditions of the planner's problem:

$$\begin{aligned} E_t^J \left[ \frac{\partial}{\partial y_t} b(y_{t-1}, y_t, y_{t+1}, u_t) + \beta L^{+1} \frac{\partial}{\partial y_{t-1}} b(y_{t-1}, y_t, y_{t+1}, u_t) \right. \\ + \beta^{-1} \lambda_{t-1}^T E_{t-1}^I \left[ L^{-1} \frac{\partial}{\partial y_{t+1}} f(y_{t-1}, y_t, y_{t+1}, u_t) \right] \\ + \lambda_t^T E_t^I \left[ \frac{\partial}{\partial y_t} f(y_{t-1}, y_t, y_{t+1}, u_t) \right] \\ \left. + \beta \lambda_{t+1}^T E_{t+1}^I \left[ L^{+1} \frac{\partial}{\partial y_{t-1}} f(y_{t-1}, y_t, y_{t+1}, u_t) \right] \right] = 0, \end{aligned} \quad (5)$$

where  $L^{+1}$  and  $L^{-1}$  are one period lead and lag operators respectively.

Now we have to make a few assertions concerning expectations conditioned on the different information sets to simplify (5). Recall the formula for integration through information on which another expectation is conditioned, this is:

$$E[E[u|v]] = E[u],$$

where the outer expectation integrates through  $v$ . Since  $J_t \subset I_t$ , by easy application of the above formula we obtain

$$\begin{aligned} E_t^J [E_t^I [X]] &= E_t^J [X] \quad \text{and} \\ E_t^J [E_{t-1}^I [X]] &= E_t^J [X] \\ E_t^J [E_{t+1}^I [X]] &= E_{t+1}^J [X] \end{aligned} \tag{6}$$

Now, the last term of (5) needs a special attention. It is equal to  $E_t^J [\beta \lambda_{t+1}^T E_{t+1}^I [X]]$ . If we assume that the problem (3) has a solution, then there is a deterministic function from  $J_{t+1}$  to  $\lambda_{t+1}$  and so  $\lambda_{t+1} \in J_{t+1} \subset I_{t+1}$ . And the last term is equal to  $E_t^J [E_{t+1}^I [\beta \lambda_{t+1}^T X]]$ , which is  $E_{t+1}^J [\beta \lambda_{t+1}^T X]$ . This term can be equivalently written as  $E_t^J [\beta \lambda_{t+1}^T E_{t+1}^J [X]]$ . The reason why we write the term in this way will be clear later. All in all, we have

$$\begin{aligned} E_t^J \left[ \frac{\partial}{\partial y_t} b(y_{t-1}, y_t, y_{t+1}, u_t) + \beta L^{+1} \frac{\partial}{\partial y_{t-1}} b(y_{t-1}, y_t, y_{t+1}, u_t) \right. \\ \left. + \beta^{-1} \lambda_{t-1}^T L^{-1} \frac{\partial}{\partial y_{t+1}} f(y_{t-1}, y_t, y_{t+1}, u_t) \right. \\ \left. + \lambda_t^T \frac{\partial}{\partial y_t} f(y_{t-1}, y_t, y_{t+1}, u_t) \right. \\ \left. + \beta \lambda_{t+1}^T E_{t+1}^J \left[ L^{+1} \frac{\partial}{\partial y_{t-1}} f(y_{t-1}, y_t, y_{t+1}, u_t) \right] \right] = 0. \end{aligned} \tag{7}$$

Note that we have not proved that (5) and (7) are equivalent. We proved only that if (5) has a solution, then (7) is equivalent (and has the same solution).