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Taking Perturbation to the Accuracy Frontier: A Hybrid of Local and Global Solutions*

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Abstract

Perturbation methods produce solutions of lower accuracy than global Euler equation-based methods. In the present paper, we implement a hybrid method that solves for some policy functions locally (using standard perturbation) and solves for the other policy functions globally (using closed-form expressions and a numerical solver). Our hybrid method extends the current speed-accuracy frontier: for a multi-country RBC model used for comparing numerical methods in a special 2011 issue of the JEDC, we attain higher accuracy of solutions than any other method participating in the comparison analysis. Our solutions are computed with the help of Dynare, and the programs are publicly available.

JEL classification: C63; C68; C88; F41

Keywords: Dynare; Perturbation; Hybrid; Accuracy; Numerical methods; Approximation

1 Introduction

Perturbation methods are widely used in the literature for solving, estimating and simulating dynamic economic models.¹ Such methods compute solutions around the deterministic steady state using Taylor expansions of optimality conditions. The advantage of perturbation methods is their low computational expense. The main drawback of these methods is that the accuracy of local solutions decreases rapidly away from the steady state; see [Judd and Guu \(1993\)](#) and [Kollmann et al. \(2011b\)](#) for accuracy results in a deterministic one-agent and stochastic multi-agent economies, respectively. In particular, the first- and second-order perturbation methods of [Kollmann et al. \(2011a\)](#) produce maximum approximation errors of 6.3% and 1.35% on the ergodic set, and they produce maximum errors of 65% and 50% on a 30% deviation from the steady state; see [Kollmann et al. \(2011b\)](#). In turn, global methods such as projection and stochastic simulation methods compute solutions on large domains (such as deterministic grids

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¹Perturbation methods are studied in [Judd and Guu \(1993\)](#), [Gaspar and Judd \(1997\)](#), [Judd \(1998\)](#), [Collard and Juillard \(2001\)](#), [Jin and Judd \(2002\)](#) and [Schmitt-Grohé and Uribe \(2004\)](#), among others.

of points or simulated series), and the range of their accuracy is much wider. In the accuracy comparison of [Kollmann et al. \(2011b\)](#), errors produced by perturbation and global solution methods can differ by up to three orders of magnitude.²

In the present paper, we implement a hybrid method that solves for some policy functions locally (using standard perturbation) and solves for the remaining policy functions globally (using closed-form expressions and a numerical solver). The hybrid solution is not uniquely defined: considering different combinations of local and global policy functions and different combinations of optimality conditions leads to distinct hybrid solutions. We describe the accuracy and cost considerations that can guide us in the choice of a specific hybrid solution. We also illustrate the construction of hybrid solutions by way of examples.

We assess the performance of the proposed hybrid solution method in the context of the models studied in the comparison analysis of [Kollmann et al. \(2011b\)](#). We compute perturbation solutions up to the third-order approximation using Dynare (note that in the comparison analysis of [Kollmann et al. \(2011b\)](#), at most second-order approximations are studied). We implement a specific hybrid method that takes perturbation policy functions for the endogenous state variables and computes the remaining variables to satisfy the optimality conditions exactly. By construction, such a hybrid solution has errors only in Euler equations; errors in all other equations are zeros (up to the given degree of accuracy). Our hybrid method extends the current speed-accuracy frontier: we attain higher accuracy of solutions than any other method participating in the comparison analysis of [Kollmann et al. \(2011b\)](#). In particular, the maximum error produced by the hybrid method on the ergodic set in the eight-country version of the model does not exceed 0.001%, which is three times smaller than the corresponding statistics attained by a cluster grid algorithm (CGA) of [Maliar et al. \(2011\)](#) (the most accurate on the ergodic set). For models with a larger number of countries, increases in accuracy are even more sizable.

Our general presentation of the hybrid method encompasses some particular cases already studied in the literature. First, [Dotsey and Mao \(1992\)](#) compare the performance of a standard linearization method with a specific hybrid method that uses a linearized policy function for capital and labor and solves for consumption and investment analytically from the optimality conditions. In the context of an RBC model with labor and production taxes, none of the studied methods strictly dominates the other: the hybrid method produces more accurate policy functions for investment, while the linearization method produces more accurate policy functions for consumption and real interest rates. Furthermore, [Maliar et al. \(2010, 2011\)](#) extend the hybrid method to include those cases in which optimality conditions do not admit closed-form solutions. They construct a hybrid solution that combines log-linear policy functions (used to find capital) and fixed point “iteration-on-allocation” numerical solver (used to solve for consumption and labor allocations) in the context of models studied in [Kollmann et al. \(2011b\)](#). The hybrid method delivers accuracy levels that are of more than an order of magnitude higher than those produced by the standard log-linearization method.

The rest of the paper is organized as follows. Section 2 presents the hybrid method that combines local (perturbation) and global solutions. Section 3 assesses hybrid solutions in the context of one of the RBC models studied in [Kollmann et al. \(2011b\)](#). Section 4 concludes.

²In a special 2011 issue of the JEDC, [Kollmann et al. \(2011b\)](#) compare the performance of the following six methods: first- and second-order perturbation methods of [Kollmann et al. \(2011a\)](#), stochastic simulation and cluster-grid algorithms of [Maliar et al. \(2011\)](#), Smolyak’s collocation method of [Malin et al. \(2011\)](#), and a monomial rule Galerkin method of [Pichler \(2011\)](#). The latter three methods are projection methods. The accuracy evaluation is performed using the software developed by [Juillard and Villemot \(2011\)](#).

2 Presentation of the hybrid method

In this section, we describe the studied class of problems, outline the standard perturbation method, elaborate a hybrid method that combines local and global solutions, analyze the determinants of accuracy and speed of the methods considered, and illustrate the construction of hybrid solutions using the example of the standard neoclassical growth model.

2.1 The studied class of problems

We study a class of problems, whose solutions are characterized by the following set of equations for $t = 0, 1, \dots$

$$E_t [H(\mathbf{x}_t, \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{z}_{t+1}, \mathbf{y}_{t+1})] = \mathbf{0}, \quad (1)$$

$$G(\mathbf{x}_t, \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{t+1}) = \mathbf{0}, \quad (2)$$

$$\mathbf{z}_{t+1} = \Phi \mathbf{z}_t + \boldsymbol{\varepsilon}_{t+1}, \quad (3)$$

where initial condition $(\mathbf{x}_0, \mathbf{z}_0)$ is given; E_t denotes the operator of conditional expectation; $\mathbf{x}_t \in \mathbb{R}^{n_x}$ is a vector of endogenous state variables at t (*e.g.*, capital); $\mathbf{z}_t \in \mathbb{R}^{n_z}$ is a vector of exogenous state (random) variables at t (*e.g.*, productivity); $\mathbf{y}_t \in \mathbb{R}^{n_y}$ is a vector that contains t -period control variables (*e.g.*, consumption, labor) and other variables (*e.g.*, prices, Lagrange multipliers) given by functions of variables known at t ; with a slight abuse of terminology, we refer to \mathbf{y}_t as control variables; $\mathbf{0}$ denotes a column-vector of zeros; H and G are vector functions that are assumed to be q -times continuously differentiable; $\boldsymbol{\varepsilon}_{t+1} \in \mathbb{R}^{n_z}$ is a vector of disturbances, $\boldsymbol{\varepsilon}_{t+1} \sim \mathcal{N}(\mathbf{0}, \Sigma)$ with $\Sigma \in \mathbb{R}^{n_z \times n_z}$ being a variance-covariance matrix; $\Phi \in \mathbb{R}^{n_z \times n_z}$ determines the auto-correlation of \mathbf{z}_t and has eigenvalues whose absolute values are less than one.³

Equations of type (1) include variables not known at t , \mathbf{z}_{t+1} and \mathbf{y}_{t+1} , and require the evaluation of conditional expectations. We refer to these n_H equations as *inter-temporal-choice conditions*. Equations of type (2) contain only variables known at t . We refer to these n_G equations as *intra-temporal-choice conditions*.⁴ Finally, equations of type (3) are laws of motion for exogenous state variables; there are n_z such equations.

A deterministic steady state is defined as a set of values for the endogenous variables, $\mathbf{x}_t = \mathbf{x}_{t+1} = \bar{\mathbf{x}}$ and $\mathbf{y}_t = \mathbf{y}_{t+1} = \bar{\mathbf{y}}$, that solves equations (1) and (2) in the absence of exogenous shocks (*i.e.*, $\mathbf{z}_t = \mathbf{z}_{t+1} = \mathbf{0}$, $H(\bar{\mathbf{x}}, \mathbf{0}, \bar{\mathbf{y}}, \bar{\mathbf{x}}, \mathbf{0}, \bar{\mathbf{y}}) = \mathbf{0}$ and $G(\bar{\mathbf{x}}, \mathbf{0}, \bar{\mathbf{y}}, \bar{\mathbf{x}}) = \mathbf{0}$). Let $\mathbf{e}_t \equiv \{\mathbf{x}_{t+1}, \mathbf{y}_t\}$ be the vector of all endogenous variables whose value is determined at t (namely, current-period control and next-period state variables). A solution is defined as a set of policy functions for endogenous variables $\mathbf{e}_t = \Psi(\mathbf{x}_t, \mathbf{z}_t)$ that satisfy the optimality conditions (1) and (2) in the relevant region of the state space. Note that the total number of policy functions, $n \equiv n_x + n_y$, is equal to the total number of equations in the system (1) and (2), that is $n_H + n_G$. We assume that functions H and G satisfy jointly a set of regularity conditions that are sufficient to ensure that a steady state exists and is unique and that a solution exists and is unique.

2.2 Standard perturbation method

The standard perturbation method approximates policy functions for endogenous variables \mathbf{e}_t around one point of the state space (the deterministic steady state) using a Taylor expansion of the system of equations (1) and (2). The perturbation method of order p implements a p -th

³Alternatively, \mathbf{x}_t and \mathbf{z}_t can be called, respectively, endogenous and exogenous predetermined variables at the beginning of period t , and \mathbf{y}_t can be called endogenous non-predetermined variables.

⁴Note that \mathbf{x}_{t+1} is t -measurable and therefore, there is no conditional expectation in equations of type (2).

order Taylor expansion. We denote by $\widehat{\Psi}(\mathbf{x}_t, \mathbf{z}_t)$ approximate policy functions delivered by the perturbation method.

The main advantage of the standard perturbation method is that its computational expense is typically low even for problems with high dimensionality. The drawback is that the local solutions delivered by the perturbation method are accurate only in a small neighborhood of the steady state; the accuracy decreases rapidly away from the steady state point around which the solution is computed. See [Judd and Guu \(1993\)](#) for the accuracy results in a deterministic one-agent economy and [Kollmann et al. \(2011b\)](#) for the accuracy results in stochastic multi-agent economies.

The difference in accuracy between the perturbation and global solution methods is not very large near the steady state, however it increases rapidly away from the steady state. In particular, for a 30% deviation from the steady state, the errors in some optimality conditions amount to 65% for the first-order and 50% for the second-order perturbation methods of [Kollmann et al. \(2011a\)](#) (see [Kollmann et al., 2011b](#), Table 4, Model III). Even on the ergodic set (stochastic simulation), errors can be as large as 6.3% and 1.35% for the first- and second-order perturbation methods, respectively (see [Kollmann et al., 2011b](#), Table 5, Model III). These accuracy levels are not sufficient for many economic applications. For comparison, the second-degree polynomial approximations delivered by CGA of [Maliar et al. \(2011\)](#)—the most accurate on the ergodic set—has errors of at most 0.009% (see [Kollmann et al., 2011b](#), Table 5, Model IV).

2.3 Hybrid of local and global solutions

The purpose of this section is to construct a perturbation-based method that can deliver accuracy levels comparable to those of global solution methods. We specifically develop a hybrid method that solves for some policy functions locally (using standard perturbation) and solves for the other policy functions globally (using closed-form expressions and a numerical solver). This method is as follows:

- Step 1. Compute a standard perturbation solution and partition the n policy functions into two groups, $\widehat{\Psi}(\mathbf{x}_t, \mathbf{z}_t) \equiv \left\{ \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t), \widehat{\Psi}^2(\mathbf{x}_t, \mathbf{z}_t) \right\}$. The perturbation policy functions $\widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)$ will be used for constructing a hybrid solution (let n^1 be the total number of such policy functions, $0 \leq n^1 \leq n$). The remaining perturbation policy functions $\widehat{\Psi}^2(\mathbf{x}_t, \mathbf{z}_t)$ will be replaced with a global solution (the number of such policy functions is $n^2 \equiv n - n^1$).
- Step 2. Partition the system of n equations (1) and (2) into two sub-systems with n^1 and n^2 equations. We assume that the second sub-system with n^2 equations is non-degenerate and identifies n^2 policy functions $\Psi^2(\mathbf{x}_t, \mathbf{z}_t)$ uniquely if $\Psi^1(\mathbf{x}_t, \mathbf{z}_t)$ is given.
- Step 3. Given the perturbation policy functions $\widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)$ chosen in Step 1, solve (either analytically or using a numerical solver) for the n^2 policy functions $\Psi^2(\mathbf{x}_t, \mathbf{z}_t; \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t))$ that satisfy the n^2 equations chosen in Step 2.

The hybrid solution is $\widetilde{\Psi}(\mathbf{x}_t, \mathbf{z}_t) \equiv \left\{ \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t), \Psi^2(\mathbf{x}_t, \mathbf{z}_t; \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)) \right\}$.

The above description makes it clear that a hybrid solution is not uniquely defined. First, in Step 1, there are many ways of choosing how many and which perturbation policy functions to keep. Second, in Step 2, there are many ways of choosing which optimality conditions to use for identifying the remaining policy functions. In particular, two obvious limiting cases

of the hybrid method are to keep the original perturbation solution unchanged ($n^1 = n$) and to re-compute the entire solution globally ($n^1 = 0$). Below, we discuss the cost and accuracy considerations that guide us in the choice of a specific hybrid solution.

2.4 Cost considerations

If $\Psi^2(\mathbf{x}_t, \mathbf{z}_t; \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t))$ can be constructed analytically using the chosen set of the optimality conditions, the cost of finding a hybrid solution is approximately the same as the cost of finding a perturbation solution. The only difference between the two solutions is that we use different formulas for computing some endogenous variables.

If $\Psi^2(\mathbf{x}_t, \mathbf{z}_t; \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t))$ cannot be constructed analytically, the cost of finding a hybrid solution can be substantially higher than that of finding a perturbation solution. To reduce the cost of constructing hybrid solutions, one must choose a set of optimality conditions which are the least expensive to solve (typically, it is easier to solve equations of type (2) than equations of type (1) as the former type does not require the evaluation of conditional expectations).

2.5 Accuracy considerations

Consider the standard perturbation solution and construct an accuracy measure that reflects the difference between the approximate policy functions $\widehat{\Psi}(\mathbf{x}_t, \mathbf{z}_t)$ and the true policy functions $\Psi(\mathbf{x}_t, \mathbf{z}_t)$ within the relevant region of the state space

$$\widehat{\Delta}^i \equiv \left\| \widehat{\Psi}^i(\mathbf{x}_t, \mathbf{z}_t) - \Psi^i(\mathbf{x}_t, \mathbf{z}_t) \right\|, \quad (4)$$

where $i = 1, 2$, and $\|\cdot\|$ denotes the distance measure used. Let $\widetilde{\Delta}^i$, $i = 1, 2$ be the corresponding accuracy measure for a hybrid solution. Note that by construction of the hybrid solution, $\widetilde{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t) = \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)$ and hence, $\widetilde{\Delta}^1 = \widehat{\Delta}^1$. This implies that a hybrid solution is more (less) accurate than the perturbation solution whenever $\widehat{\Delta}^2 > \widetilde{\Delta}^2$ ($\widehat{\Delta}^2 < \widetilde{\Delta}^2$). We now compare the accuracy of the standard perturbation and hybrid solutions in two limiting cases.

Claim.

- a) If $\widehat{\Delta}^1 = 0$ and $\widehat{\Delta}^2 > 0$, then any hybrid solution is more accurate than the perturbation solution.
- b) If $\widehat{\Delta}^1 > 0$ and $\widehat{\Delta}^2 = 0$, then any hybrid solution is less accurate than the perturbation solution.

Proof.

- a) A zero error $\widehat{\Delta}^1 = 0$ means that the corresponding perturbation policy functions coincide with the true solution, *i.e.*, $\widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t) = \Psi^1(\mathbf{x}_t, \mathbf{z}_t)$. Since the remaining hybrid policy functions $\widetilde{\Psi}^2(\mathbf{x}_t, \mathbf{z}_t)$ are constructed under the true solution $\Psi^1(\mathbf{x}_t, \mathbf{z}_t)$, the hybrid solution coincides with the true solution $\widetilde{\Psi}^2(\mathbf{x}_t, \mathbf{z}_t; \Psi^1(\mathbf{x}_t, \mathbf{z}_t)) = \Psi^2(\mathbf{x}_t, \mathbf{z}_t)$, and hence, $\widetilde{\Delta}^2 = 0$.
- b) A non-zero error $\widehat{\Delta}^1 > 0$ means that some policy functions used for constructing a hybrid solution do not coincide with the true solution, *i.e.*, $\widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t) \neq \Psi^1(\mathbf{x}_t, \mathbf{z}_t)$. Since the remaining hybrid policy functions $\Psi^2(\mathbf{x}_t, \mathbf{z}_t; \widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t))$ are constructed using policy functions that do not coincide with the true solution $\widehat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t) \neq \Psi^1(\mathbf{x}_t, \mathbf{z}_t)$, and since

the true solution is by assumption unique, the hybrid solution does not coincide with the true solution $\tilde{\Psi}^2(\mathbf{x}_t, \mathbf{z}_t; \hat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)) \neq \Psi^2(\mathbf{x}_t, \mathbf{z}_t)$ and hence, $\tilde{\Delta}^2 > 0$.

□

These results show that the accuracy of a hybrid solution is entirely determined by the accuracy of the perturbation policy functions $\hat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)$ used for its construction. Small (large) errors in the perturbation policy functions induce small (large) errors in the hybrid solution. Reducing the number of perturbation policy functions in $\hat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)$ will not necessarily lead to more accurate hybrid solutions. In particular, if a hybrid solution is constructed using only one perturbation policy function but with large errors, all the constructed hybrid policy functions will inherit these large errors.

In general, all policy functions delivered by a perturbation method have non-negligible errors. Hence, a ranking of perturbation and hybrid solutions cannot be generally established; such a ranking depends on the given model and the specific construction of a hybrid solution. Nonetheless, the above results provide us with a guideline in the choice of an accurate hybrid solution: we must choose those policy functions for constructing hybrid solutions, $\hat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t)$, that are computed most accurately by the standard perturbation method. Furthermore, to identify the remaining policy functions, $\Psi^2(\mathbf{x}_t, \mathbf{z}_t; \hat{\Psi}^1(\mathbf{x}_t, \mathbf{z}_t))$, we must use those optimality conditions from the system (1) and (2) that have the largest approximation errors under the original perturbation solution (in order to reduce such errors).

2.6 An illustration: one-sector growth model

We now illustrate the construction of hybrid solutions using the standard one-sector neoclassical growth model.

$$\max_{\{k_{t+1}, c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (5)$$

$$\text{s. t. } c_t + k_{t+1} = k_t + a_t f(k_t), \quad (6)$$

$$\ln a_{t+1} = \rho \ln a_t + \varepsilon_{t+1}, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2), \quad (7)$$

where c_t , k_t and a_t are, respectively, consumption, capital and productivity level at period t ; u and f are, respectively, the utility and production functions, which are assumed to be increasing and concave; $a_t f(k_t)$ is net output (*i.e.*, output minus depreciated capital); $\beta \in (0, 1)$ is the discount factor; $\rho \in (0, 1)$ and $\sigma \geq 0$ are the auto-correlation coefficient and standard deviation of the productivity level, respectively; initial condition (k_0, a_0) is given. In terms of the notation of section 2, k_t , a_t and c_t correspond to \mathbf{x}_t , \mathbf{z}_t and \mathbf{y}_t , respectively. The Euler equation of problem (5)–(7) is

$$u'(c_t) = \beta E_t \{u'(c_{t+1}) a_{t+1} f'(k_{t+1})\}. \quad (8)$$

A solution to problem (5)–(7) is given by policy functions for capital $k_{t+1} = K(k_t, a_t)$ and consumption $c_t = C(k_t, a_t)$ satisfying (8) and (6).

Four hybrid solutions. The standard perturbation method delivers approximate policy functions for capital $k_{t+1} = \hat{K}(k_t, a_t)$ and consumption $c_t = \hat{C}(k_t, a_t)$. Furthermore, we have two optimality conditions, budget constraint (6) and Euler equation (8). By considering different combinations of the two policy functions and the two optimality conditions, we construct four different hybrid solutions.

HYB1: Fix $\widehat{K}(k_t, a_t)$ and define $\widetilde{C}(k_t, a_t) = c_t$ from (6),

$$c_t = k_t + a_t f(k_t) - \widehat{K}(k_t, a_t).$$

HYB2: Fix $\widehat{K}(k_t, a_t)$ and define $\widetilde{C}(k_t, a_t)$ from (8),

$$u'(\widetilde{C}(k_t, a_t)) = \beta E_t \left\{ u' \left[\widetilde{C}(\widehat{K}(k_t, a_t), a_{t+1}) \right] a_{t+1} f'(\widehat{K}(k_t, a_t)) \right\},$$

where $a_{t+1} = a_t^\rho \exp(\varepsilon_{t+1})$.

HYB3: Fix $\widehat{C}(k_t, a_t)$ and define $\widetilde{K}(k_t, a_t) = k_{t+1}$ from (6),

$$k_{t+1} = k_t + a_t f(k_t) - \widehat{C}(k_t, a_t).$$

HYB4: Fix $\widehat{C}(k_t, a_t)$ and define $\widetilde{K}(k_t, a_t) = k_{t+1}$ from (8),

$$u'(\widehat{C}(k_t, a_t)) = \beta E_t \left\{ u'(\widehat{C}(k_{t+1}, a_{t+1})) a_{t+1} f'(k_{t+1}) \right\}.$$

The cost of constructing the above hybrid solutions differs. HYB1 and HYB3 are obtained using simple closed-form expressions. HYB2 has the same cost as a global Euler-equation method (in this specific example there is no gain in approximating the capital policy function with perturbation compared to global solution methods). Finally, computing HYB4 has a small cost—the same as perturbation—but simulating this solution has a high cost: the capital choice k_{t+1} is defined implicitly and must be computed with a numerical solver in each simulated point.

As far as accuracy is concerned, the standard perturbation solution has non-zero approximation errors in both the Euler equation and budget constraint. In turn, a hybrid solution by construction has non-zero errors in just one of the two optimality conditions (namely, HYB1 and HYB3 have zero errors in budget constraint (6), and HYB2 and HYB4 have zero errors in Euler equation (8)). We will not perform a numerical analysis of equilibrium in this simple model but will move to applications that pose a challenge to the recent literature on numerical methods.

3 Assessing hybrid solutions: a multi-country RBC model

A special 2011 issue of the JEDC compares the performance of six state-of-art numerical methods using a collection of multi-country RBC models; see [Kollmann et al. \(2011b\)](#) for the comparison results. We assess the performance of the hybrid method in the context of a multi-country model used in the above comparison analysis.

3.1 The model

In the description of the model, we follow [Juillard and Villemot \(2011\)](#). The world economy consists of a finite number of countries N , and each country is populated by a representative consumer. A social planner maximizes a weighted sum of expected lifetime utility of the countries' representative consumers

$$\max_{\{c_t^j, \ell_t^j, i_t^j, k_{t+1}^j\}_{t=0, \dots, \infty}^{j=1, \dots, N}} E_0 \sum_{j=1}^N \tau^j \left(\sum_{t=0}^{\infty} \beta^t u^j(c_t^j, \ell_t^j) \right) \quad (9)$$

subject to the world resource constraint,

$$\sum_{j=1}^N \left(c_t^j + i_t^j - \delta k_t^j \right) = \sum_{j=1}^N \left(a_t^j f^j \left(k_t^j, \ell_t^j \right) - \frac{\phi}{2} k_t^j \left(\frac{i_t^j}{k_t^j} - 1 \right)^2 \right), \quad (10)$$

and to the capital-accumulation equations for $j = 1, \dots, N$,

$$k_{t+1}^j = (1 - \delta) k_t^j + i_t^j, \quad (11)$$

where $c_t^j, \ell_t^j, i_t^j, k_t^j, a_t^j, u^j, f^j$ and τ^j are, respectively, consumption, labor, investment, capital, productivity level, utility function, net production function and welfare weight of a country $j = 1, \dots, N$; ϕ is the adjustment-cost parameter. Initial condition $(\mathbf{k}_0, \mathbf{a}_0)$ is given, where $\mathbf{k}_t \equiv (k_t^1, \dots, k_t^N)$ and $\mathbf{a}_t \equiv (a_t^1, \dots, a_t^N)$. The process for the productivity level in country j is given by

$$\ln a_t^j = \rho \ln a_{t-1}^j + \sigma \varepsilon_t^j, \quad (12)$$

where $\varepsilon_t^j \equiv e_t + e_t^j$ with e_t and e_t^j being common and country-specific productivity shocks, respectively, and $e_t, e_t^j \sim \mathcal{N}(0, 1)$; ρ is the auto-correlation coefficient of the productivity level; and σ determines the standard deviation of the productivity level.

An interior solution to the social planner's problem (9)–(12) satisfies the following optimality conditions for countries $j = 1, \dots, N$

$$\tau^j u_c^j \left(c_t^j, \ell_t^j \right) = \lambda_t, \quad (13)$$

$$\tau^j u_\ell^j \left(c_t^j, \ell_t^j \right) = -\lambda_t a_t^j f_\ell^j \left(k_t^j, \ell_t^j \right), \quad (14)$$

$$\lambda_t \left[1 + \phi \left(\frac{i_t^j}{k_t^j} - \delta \right) \right] = \beta E_t \left\{ \lambda_{t+1} \left[1 + a_{t+1}^j f_k^j \left(k_{t+1}^j, \ell_{t+1}^j \right) + \phi \left(1 - \delta + \frac{i_{t+1}^j}{k_{t+1}^j} - \frac{1}{2} \left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right) \left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right] \right\}, \quad (15)$$

where λ_t is the Lagrange multiplier associated with world resource constraint (10). Here, and for the remainder of the paper, notation of type F_{χ_m} stands for the first-order partial derivative of a function $F(\dots, \chi_m, \dots)$ with respect to a variable χ_m . In addition, the social planner's solution satisfies the set of transversality conditions $\lim_{t \rightarrow \infty} \lambda_t k_{t+1}^j = 0$ for $j = 1, \dots, N$.

3.2 A specific hybrid solution

In this section, we describe a specific version of the hybrid method of section 2.3 that will be used for solving the model (9)–(12). We assume that the system (1) and (2) has the same number of inter-temporal-choice optimality conditions as the number of endogenous state variables, $n_H = n_x$, and that it has the same number of inter-temporal-choice optimality conditions as the number of control variables, $n_G = n_y$. Note that, in particular, this assumption is satisfied for the model (9)–(12) described in section 3.1. We proceed as follows:

- Step 1. Compute a standard perturbation solution and partition the perturbation policy functions into those for the next-period state variables and those for the control variables,

$\widehat{\Psi}(\mathbf{x}_t, \mathbf{z}_t) \equiv \left\{ \widehat{X}(\mathbf{x}_t, \mathbf{z}_t), \widehat{Y}(\mathbf{x}_t, \mathbf{z}_t) \right\}$. The perturbation policy functions for the next-period state variables $\mathbf{x}_{t+1} = \widehat{X}(\mathbf{x}_t, \mathbf{z}_t)$ will be used for constructing a hybrid solution (the number of such policy functions is n_x). The perturbation policy functions for the control variables $\mathbf{y}_t = \widehat{Y}(\mathbf{x}_t, \mathbf{z}_t)$ will be replaced with a global solution (the number of such policy functions is n_y).

- Step 2. Partition the system of n equations (1) and (2) into two sub-systems, one with n_x inter-temporal-choice conditions (1) and the other with n_y intra-temporal-choice conditions (2). Substitute the perturbation policy functions $\mathbf{x}_{t+1} = \widehat{X}(\mathbf{x}_t, \mathbf{z}_t)$ in (2) to obtain

$$G(\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t, \widehat{X}(\mathbf{x}_t, \mathbf{z}_t)) = \mathbf{0}. \quad (16)$$

- Step 3. Given n_x perturbation policy functions for the next-period state variables $\widehat{X}(\mathbf{x}_t, \mathbf{z}_t)$, solve (either analytically or using a numerical solver) for n_y policy functions

$$\mathbf{y}_t = \widetilde{Y}(\mathbf{x}_t, \mathbf{z}_t; \widehat{X}(\mathbf{x}_t, \mathbf{z}_t))$$

satisfying the n_y intra-temporal-choice conditions (16).

The hybrid solution is $\widetilde{\Psi}(\mathbf{x}_t, \mathbf{z}_t) \equiv \left\{ \widehat{X}(\mathbf{x}_t, \mathbf{z}_t), \widetilde{Y}(\mathbf{x}_t, \mathbf{z}_t; \widehat{X}(\mathbf{x}_t, \mathbf{z}_t)) \right\}$.

In terms of our example of section 2.6, this hybrid solution corresponds to HYB1. In terms of the model (9)–(12), the original perturbation solution consists of $4N + 1$ policy functions $k_{t+1}^j = \widehat{K}^j(\mathbf{k}_t, \mathbf{a}_t)$, $c_t^j = \widehat{C}^j(\mathbf{k}_t, \mathbf{a}_t)$, $\ell_t^j = \widehat{L}^j(\mathbf{k}_t, \mathbf{a}_t)$ and $i_t^j = \widehat{I}^j(\mathbf{k}_t, \mathbf{a}_t)$, $j = 1, \dots, N$ and $\lambda_t = \widehat{\Lambda}(\mathbf{k}_t, \mathbf{a}_t)$ computed using the $4N + 1$ optimality conditions (10), (11), (13)–(15). To construct the hybrid solution, we take N perturbation policy functions for the next-period capital $k_{t+1}^j = \widetilde{K}^j(\mathbf{k}_t, \mathbf{a}_t)$ and compute the remaining $3N + 1$ policy functions $c_t^j = \widetilde{C}^j(\mathbf{k}_t, \mathbf{a}_t)$, $\ell_t^j = \widetilde{L}^j(\mathbf{k}_t, \mathbf{a}_t)$, $i_t^j = \widetilde{I}^j(\mathbf{k}_t, \mathbf{a}_t)$, $j = 1, \dots, N$ and $\lambda_t = \widetilde{\Lambda}(\mathbf{k}_t, \mathbf{a}_t)$ to satisfy the intra-temporal-choice conditions (10), (11), (13) and (14)⁵ (thus, N Euler equations (15) are not used for constructing this specific hybrid solution).

The above construction is motivated by the following accuracy and cost considerations in the context of the given model. First, Maliar et al. (2011) show that one must approximate more accurately the consumption and labor policy functions than the capital policy functions to attain a target accuracy level. Second, Kollmann et al. (2011b) report that the perturbation methods produce smaller errors in inter-temporal-choice conditions (Euler equations) (15) than in the intra-temporal-choice conditions (10), (11), (13) and (14). Third, Maliar et al. (2011) argue that fixing the capital policy functions helps save on cost because one can first simulate the path for the state variables $\{\mathbf{k}_{t+1}, \mathbf{a}_t\}_{t=1, \dots, T}$ without solving for the remaining variables $\{\mathbf{c}_t, \ell_t, \mathbf{i}_t, \lambda_t\}_{t=1, \dots, T}$ at each date and later fill in such remaining variables, where $\mathbf{c}_t \equiv (c_t^1, \dots, c_t^N)$, $\ell_t \equiv (\ell_t^1, \dots, \ell_t^N)$, $\mathbf{i}_t \equiv (i_t^1, \dots, i_t^N)$. Finally, disregarding Euler equations (15) helps save on cost because it allows us to avoid expensive multi-dimensional integration.

3.3 An example with a closed-form solution for control variables

We illustrate the construction of the hybrid solution described in section 3.2 using a version of the model (9)–(12) in which control variables can be characterized analytically in terms of state variables. We specifically assume that leisure is non-valued and that the utility function is

⁵Note that λ_t is not a control variable; it is a function of variables known at t (recall that we augmented \mathbf{y}_t to include variables like λ_t).

logarithmic, *i.e.*, $u^j(c_t^j, \ell_t^j) = \ln c_t^j$. This setup corresponds to Model I in [Juillard and Villemot \(2011\)](#).

We take the capital policy functions $k_{t+1}^j = \widehat{K}^j(\mathbf{k}_t, \mathbf{a}_t)$, $j = 1, \dots, N$ delivered by the standard perturbation method and compute the remaining policy functions to satisfy (10), (11) and (13). To be specific, we first find the investment policy function $\widetilde{I}_t^j(\mathbf{k}_t, \mathbf{a}_t) = i_t^j$, $j = 1, \dots, N$,

$$i_t^j = \widehat{K}^j(\mathbf{k}_t, \mathbf{a}_t) - (1 - \delta) k_t^j. \quad (17)$$

Given $\widetilde{I}_t^j(\mathbf{k}_t, \mathbf{a}_t)$, we compute aggregate consumption $c_t \equiv \sum_{j=1}^N c_t^j$,

$$c_t = \sum_{j=1}^N \left(a_t^j f^j(k_t^j, 1) - \frac{\phi}{2} k_t^j \left(\frac{\widetilde{I}_t^j(\mathbf{k}_t, \mathbf{a}_t)}{k_t^j} - 1 \right)^2 - \widetilde{I}_t^j(\mathbf{k}_t, \mathbf{a}_t) + \delta k_t^j \right). \quad (18)$$

Condition (13) implies $\frac{\tau^j}{c_t^j} = \lambda_t$, and hence aggregate consumption is $c_t = \sum_{j=1}^N \frac{\tau^j}{\lambda_t}$. The latter condition implies the policy function for the Lagrange multiplier, $\widetilde{\Lambda}(\mathbf{k}_t, \mathbf{a}_t) = \lambda_t$,

$$\lambda_t = \frac{1}{c_t} \sum_{j=1}^N \tau^j. \quad (19)$$

Finally, the policy functions for individual consumption, $\widetilde{C}_t^j(\mathbf{k}_t, \mathbf{a}_t) = c_t^j$, $j = 1, \dots, N$, are

$$c_t^j = \frac{\tau^j}{\sum_{j=1}^N \tau^j} c_t. \quad (20)$$

By construction, policy functions $\widetilde{C}_t^j(\mathbf{k}_t, \mathbf{a}_t)$, $\widetilde{I}_t^j(\mathbf{k}_t, \mathbf{a}_t)$, $j = 1, \dots, N$ and $\lambda_t = \widetilde{\Lambda}(\mathbf{k}_t, \mathbf{a}_t)$ satisfy optimality conditions (10), (11) and (13) exactly (*i.e.*, approximation errors in these conditions are zeros).⁶

3.4 Solving for control variables numerically

In general, the system of intra-temporal-choice conditions (16) does not admit a closed-form solution. Thus, the policy functions for the control variables must be constructed numerically (this is particularly true for a general specification of the model (9)–(12)). This can be done by using the following two approaches.

3.4.1 Approach 1: Find control variables using a numerical solver

This approach does not construct policy functions explicitly but solves directly for control variables, \mathbf{y}_t , satisfying intra-temporal-choice conditions (16). This is done for each point of the state space $(\mathbf{k}_t, \mathbf{a}_t)$ in which the solution is needed for applications (*e.g.*, for forecasting, estimating, accuracy testing). In principle, this approach can be implemented using a standard Newton-type numerical solver, though the cost of such a solver can be prohibitively high, especially in large-scale applications.

[Maliar et al. \(2011\)](#) design a simple derivative-free numerical solver “iteration-on-allocations” that can be an efficient alternative to Newton’s method in the given context. This solver defines a mapping $\widetilde{\mathbf{y}}_t = g(\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t; \widehat{X}(\mathbf{x}_t, \mathbf{z}_t))$ whose fixed point is an exact solution to (16),

⁶[Maliar and Maliar \(2003\)](#) show how to use analytical aggregation to derive similar policy functions for control variables in certain classes of heterogeneous-agent models with endogenous labor-leisure choice and agent-specific correlated shocks.

i.e., $\mathbf{y}_t^* = g\left(\mathbf{x}_t, \mathbf{y}_t^*, \mathbf{z}_t; \widehat{X}(\mathbf{x}_t, \mathbf{z}_t)\right)$, and finds the fixed point using fixed point iteration with damping, $\varsigma \tilde{\mathbf{y}}_t + (1 - \varsigma) \mathbf{y}_t$, where $\varsigma \in (0, 1)$ is a damping parameter.⁷ Under this approach, we have errors only in inter-temporal-choice conditions (1); errors in the other conditions are zero by construction (up to the target accuracy level). The cost of a vectorized version of iteration-on-allocation is low.⁸

3.4.2 Approach 2: Construct policy functions on a grid

This approach constructs policy functions for control variables on a grid of points. Specifically, we take a grid of I values for the state variables, $\{\mathbf{x}_i, \mathbf{z}_i\}_{i=1}^I$, solve for the control variables at each grid point, $\mathbf{y}_i = Y\left(\mathbf{x}_i, \mathbf{z}_i; \widehat{X}(\mathbf{x}_i, \mathbf{z}_i)\right)$, $i = 1, \dots, I$, and interpolate the resulting set of functions on the relevant continuous domain using some interpolation method (*e.g.*, a global polynomial approximation, piecewise linear polynomial approximation, splines) to obtain $\mathbf{y} = \tilde{Y}\left(\mathbf{x}, \mathbf{z}; \widehat{X}(\mathbf{x}, \mathbf{z})\right)$.

Standard tensor-product grids are feasible only for problems with low dimensionality. To make the above approach practical for high-dimensional applications, we can restrict attention to the relevant region of the state space, the ergodic set, as is done under the precomputation method developed in Maliar et al. (2011). Specifically, Maliar et al. (2011) precompute policy functions on a stochastic simulation or a grid of clusters obtained on a stochastic simulation. The accuracy and cost of the precomputation method depends on a specific implementation (the number and placement of grid points and the choice of approximating functions).⁹

3.5 Implementation details

In this section, we describe the parametrization of the model, the solution procedure, the accuracy tests implemented and the software and hardware used.

Parametrization of the model. We apply our hybrid method to solve one of the RBC models studied in the comparison analysis of Kollmann et al. (2011b), namely, Model II with an asymmetric specification. We chose this model because it represents all challenges posed in the comparison analysis: a large number of state variables, endogenous labor-leisure choice, heterogeneity in fundamentals and the absence of closed-form expressions for next-period state and control variables.¹⁰ The utility and production functions are given by

$$u^j\left(c_t^j, \ell_t^j\right) = \frac{\left(c_t^j\right)^{1-1/\gamma^j}}{1-1/\gamma^j} - b_j \frac{\left(c_t^j\right)^{1+1/\eta^j}}{1+1/\eta^j}, \quad f^j\left(k_t^j, \ell_t^j\right) = A \left(k_t^j\right)^\alpha \left(\ell_t^j\right)^{1-\alpha},$$

where $\{\gamma^j, b_j, \eta^j\}$ are the utility-function parameters, α is the capital share in production, and A is the normalizing constant in output. To make our results comparable to Kollmann et al. (2011b), we calibrate the model as is done in Juillard and Villemot (2011). We use the following values of common-for-all-countries parameters: $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\sigma = 0.01$, $\rho = 0.95$, $\phi = 0.5$, and we assume that the country-specific utility-function parameters γ^j and

⁷See Maliar et al. (2011) for a description of how this solver can be used in the context of the problem (9)–(12).

⁸See Maliar et al. (2011) for the corresponding cost estimates.

⁹See Maliar et al. (2011) for further details on how precomputation can be used in the context of the model (9)–(12).

¹⁰Model I has a degenerate labor-leisure choice, and Models III and IV are identical to Model II up to specific assumptions about preferences and technologies. In the comparison analysis of Kollmann et al. (2011b), the implications about accuracy and speed of the studied methods are similar across Models II–IV.

η^j are uniformly distributed in the intervals $[0.25, 1]$ and $[0.1, 1]$ across countries $j = 1, \dots, N$, respectively. The steady state level for productivity is normalized to one, $\bar{a}^j = 1$. We also normalize the steady state levels of capital and labor to one, $\bar{k}^j = 1$, $\bar{\ell}^j = 1$, which implies $\bar{c}^j = A$, $\bar{i}^j = \delta$, $\bar{\lambda} = 1$ and leads to $A = \frac{1-\beta}{\alpha\beta}$, $\tau^j = u_c^j(A, 1)$ and $b^j = (1 - \alpha) A^{1-1/\gamma^j}$.

Solution procedure. We use the Dynare implementation of the standard perturbation method (PER). Dynare is a software platform for handling (*i.e.*, solving, simulating, estimating) a wide class of economic models, and in particular, RBC and dynamic stochastic general equilibrium models based on the rational expectations paradigm. Dynare is able to deliver standard perturbation solutions up to third order of approximation around the steady state.¹¹

To implement our hybrid method (HYB), we solve for the values of the control variables satisfying the system of the intra-temporal-choice conditions as described in section 3.4.1. To be more specific, we compute a sequence for investment $\{\mathbf{i}_t\}_{t=1, \dots, T}$ using (11), and we find $2N + 1$ allocations $\{\mathbf{c}_t, \boldsymbol{\ell}_t, \lambda_t\}_{t=1, \dots, T}$ satisfying (10), (13) and (14) using the iteration-on-allocation numerical solver with a target unit-free accuracy of 10^{-10} and damping parameter $\varsigma = 0.01$.

Accuracy tests. To measure the accuracy of numerical solutions, we evaluate the size of approximation errors in the $4N + 1$ optimality conditions, namely, world resource constraint (10) and optimality conditions (13), (14) and (15) for $j = 1, \dots, N$. The approximation error for each optimality condition is defined as the difference between the left and right sides of the corresponding condition divided by the left side (this makes the errors to be unit-free). We focus on the maximum absolute errors across the $4N + 1$ optimality conditions.

We compute errors on two different kinds of domain. The first is a sphere of a given radius, the accuracy test on which was introduced in Judd (1992). The second domain is a stochastic simulation, the accuracy test on which was introduced in Jin and Judd (2002). We follow Juillard and Villemot (2011) in the definition of the approximation errors and in the implementation of the two accuracy tests.

Concerning the accuracy on a sphere, we consider three different values for r , namely, $r = \{0.01, 0.10, 0.30\}$. For each r , we draw 1,000 points $(\mathbf{k}_i, \mathbf{a}_i)$ such that the Euclidean distance between each point $(\mathbf{k}_i, \mathbf{a}_i)$ and the steady state $(\mathbf{1}, \mathbf{1})$ is r , *i.e.*, $\|(\mathbf{k}_i, \mathbf{a}_i) - (\mathbf{1}, \mathbf{1})\| = r$, where $\mathbf{1} \equiv (1, \dots, 1) \in \mathbb{R}^N$, and $\|\cdot\|$ is the Euclidean norm. Depending on the value of r used, this test shows how accurate the solution is on a short, medium and long distance from the steady state.

To evaluate the accuracy on a stochastic simulation, we draw a sequence of shocks of 10,200 observations using (12), and we simulate the capital time series using the capital policy functions $k_{t+1}^j = \hat{K}^j(\mathbf{k}_t, \mathbf{a}_t)$, $j = 1, \dots, N$. We start a simulation from the steady state, $(\mathbf{k}_0, \mathbf{a}_0) = (\mathbf{1}, \mathbf{1})$. To remove the effect of initial conditions, we disregard the first 200 observations. This test shows how accurate the solution is on the ergodic set of points realized in equilibrium.

Hardware and software. We use a computing workstation with two quad-core Intel® Xeon X5460 processors (clocked at 3.16Ghz), 8Gb of RAM, and running 64-bit Debian GNU/Linux. The programs are written in C++, rely on the GNU Scientific Library¹² for mathematical routines and are publicly available.¹³

¹¹See <http://www.dynare.org> and Adjemian et al. (2011) for more details on Dynare.

¹²See <http://www.gnu.org/software/gsl/> or Galassi et al. (2003).

¹³See Sébastien Villemot's web site: <http://www.dynare.org/sebastien/>.

3.6 Numerical results

In this section, we assess the performance of the proposed hybrid method in which the capital policy functions are computed by perturbation and the remaining policy functions are computed using the iteration-on-allocation numerical solver. We compare the accuracy of this hybrid method to that of the standard perturbation method.¹⁴ For both methods, we compute perturbations of first, second and third orders. For comparison, we also provide the accuracy frontier attained in the comparison analysis of [Kollmann et al. \(2011b\)](#) (with a reference to the method that delivers the best accuracy in each case).¹⁵ Note that perturbations of the third order are significantly less computationally costly than second-degree approximations for global projection methods.

In [Table 1](#), we provide the results for the model with $N = 2$. As is seen from the table, PER produces relatively small approximation errors in the Euler equations but larger approximation errors in the intra-temporal-choice conditions [\(10\)](#), [\(13\)](#) and [\(14\)](#) (although errors in the world resource constraint are smaller than those in the other two conditions). In turn, HYB produces negligible approximation errors in the intra-temporal-choice conditions by construction. The only source of errors for HYB is approximation errors in the Euler equations. In fact, Euler-equation errors for HYB are larger than those for PER. However, such errors are still smaller than errors in the intra-temporal-choice optimality conditions for PER (an exception here is the case of first-order perturbation for the test on a sphere with a small radius, $r = 0.01$). As a consequence, HYB dominates PER in accuracy. The accuracy improvements are more pronounced for the test on a stochastic simulation than for the test on a sphere. For example, under the third-order approximation, the overall accuracy increases from 0.107% for PER to 0.002% for HYB (equivalently, by $1.5 \log_{10}$ units).

The comparison with the accuracy frontier attained in the comparison analysis of [Kollmann et al. \(2011b\)](#) shows the following tendencies. First, for the test on a sphere with a small radius, $r = 0.01$, second-order HYB is almost as accurate as (second-order) CGA, and third-order HYB is more accurate by $1.58 \log_{10}$ units than (second-order) CGA. For the test on a sphere with a larger radius of $r = 0.1$, second-order HYB is not more accurate than Smolyak’s algorithm, but third-order HYB is (recall that Smolyak’s polynomial has four times more terms than a second-degree complete polynomial). For the test on a sphere with the largest radius considered, $r = 0.3$, HYB performs worse than Smolyak’s algorithm which is not surprising given that the latter method is a global solution method. Finally, for the test on a stochastic simulation, third-order HYB performs better than CGA, but second-order HYB does not.

To check the robustness of our analysis, in [Table 2](#), we report the corresponding results for the model with $N = 8$. Overall, the tendencies are very similar to those observed in [Table 1](#). The differences in accuracy between PER and HYB are even more significant reaching almost two orders of magnitude in the test on a stochastic simulation. For the test on a sphere with a small radius, $r = 0.01$, second-order approximations produced by HYB are significantly more accurate than those produced by all other methods compared in [Kollmann et al. \(2011b\)](#). For all the other tests, third-order HYB significantly outperforms the global methods compared in [Kollmann et al. \(2011b\)](#), while second-order HYB is still not better than these global methods. The gap in accuracy between HYB and PER becomes even larger for the model with $N = 10$ (we do not report the results for this model as it is not studied in [Kollmann et al., 2011b](#))

¹⁴Our implementation of the perturbation method differs from that of [Kollmann et al. \(2011a\)](#) in that they first take the logarithms of variables before constructing a Taylor expansion, while we work with the variables in levels.

¹⁵Regarding degrees of approximation in [Kollmann et al. \(2011a\)](#), [Maliar et al. \(2011\)](#) and [Pichler \(2011\)](#) use second-degree polynomials, while [Malin et al. \(2011\)](#) use Smolyak’s polynomial that have four times more terms than a second-degree complete polynomial.

As far as the cost is concerned, the solution time reported in Tables 1 and 2 for HYB coincides with that for PER. This reflects the fact that HYB uses the perturbation solution for capital and does not construct an explicit solution for the other variables. Such a solution is obtained only in the testing procedure. Furthermore, the tables indicate that HYB requires more time to run the accuracy tests than PER. Specifically, for $N = 2$, the differences in running time between HYB and PER are larger for $N = 2$ than for $N = 8$. However, HYB is still much less expensive than any global method in the literature. The relatively high cost of hybrid solutions is largely explained by a specific implementation of the iteration-on-allocation solver, namely, by using this solver in the point-by-point manner. As mentioned above, using a vectorized version of this solver will reduce the cost of hybrid solutions dramatically.

In Table 3, we report the unit-free maximum absolute differences between the simulated time series produced by any pair of the solution methods considered. In addition to the pairwise comparisons between the first-, second- and third-order PER and HYB, we make a comparison with CGA, which presumably delivers very accurate solutions. We consider the model with $N = 2$, and present the results for consumption, labor, investment and capital of countries 1 and 2. First of all notice that as the order of approximation for PER and HYB increases, the difference between time-series solutions produced by these methods and CGA monotonically decreases. An exception is an HYB solution for capital of countries 1 and 2 in which case the difference between CGA and HYB is the smallest for second-order HYB. Furthermore, in most cases the differences between HYB and CGA solutions are smaller than between PER and CGA solutions, the sole exception being consumption for which third-order PER is closer to CGA than third-order HYB.

4 Conclusion

Perturbation methods are becoming increasingly popular. In particular, Dynare provides a simple way of computing perturbation solutions to fairly sophisticated economic models including models with a large number of state variables. Perturbation methods are fast but their accuracy levels are not sufficient for certain applications. In the comparison analysis of Kollmann et al. (2011b), the perturbation methods are the least accurate solution methods and produce errors which are up to two orders of magnitude larger than those of the global solution methods.

We describe a simple modification of the standard perturbation method that reverses the situation and takes the perturbation method to the current accuracy frontier. Our third-order hybrid solutions are more accurate than any other solution presented in Kollmann et al. (2011b). Our future objective is therefore to integrate an option for hybrid solutions in the Dynare framework. To this purpose, it is necessary to automate the process of selecting a hybrid solution so that Dynare computes different hybrid solutions, compares their accuracy and delivers the most accurate solution to the user.

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Table 1: Multi-country model with $N = 2$ countries: maximum absolute errors by equation on a sphere and on a stochastic simulation.

	1st order		2nd order		3rd order		Accuracy frontier in Kollmann et al. (2011b)
	PER	HYB	PER	HYB	PER	HYB	
Solution time	0.01	0.01	0.02	0.02	0.04	0.04	
Radius $r = 0.01$							
EulerEq	-5.91	-3.83	-6.87	-5.95	-8.41	-7.60	
MUCons	-4.39	-	-5.26	-	-6.69	-	
MULabor	-4.32	-	-5.40	-	-7.03	-	
WorResConst	-4.64	-	-6.09	-	-7.83	-	
Overall	-4.32	-3.83	-5.26	-5.95	-6.69	-7.60	-6.02 (CGA)
Time to run test	0.10	4.51	0.14	3.29	0.27	2.53	
Radius $r = 0.10$							
EulerEq	-4.07	-2.44	-4.12	-3.63	-5.49	-5.20	
MUCons	-2.39	-	-3.69	-	-4.76	-	
MULabor	-2.25	-	-3.64	-	-4.81	-	
WorResConst	-2.63	-	-3.88	-	-5.20	-	
Overall	-2.25	-2.44	-3.64	-3.63	-4.76	-5.20	-4.40 (SMOL)
Time to run test	0.10	5.02	0.14	4.13	0.27	3.42	
Radius $r = 0.30$							
EulerEq	-3.08	-1.38	-2.62	-2.03	-3.55	-3.06	
MUCons	-1.43	-	-2.16	-	-2.84	-	
MULabor	-1.11	-	-1.99	-	-2.87	-	
WorResConst	-1.52	-	-2.25	-	-3.16	-	
Overall	-1.11	-1.38	-1.99	-2.03	-2.84	-3.06	-3.29 (SMOL)
Time to run test	0.10	5.62	0.14	5.07	0.27	4.67	
Stochastic simulation							
EulerEq	-3.41	-2.22	-3.37	-3.44	-4.02	-4.69	
MUCons	-1.55	-	-2.25	-	-2.97	-	
MULabor	-1.57	-	-2.29	-	-3.03	-	
ResConst	-2.10	-	-3.18	-	-4.11	-	
Overall	-1.55	-2.22	-2.25	-3.44	-2.97	-4.69	-4.50 (CGA)
Time to run test	0.98	49.48	1.42	40.43	2.67	33.03	

Notes: For each model equation (listed in the 1st column), the table reports maximum absolute errors in \log_{10} units across countries and test points. For panels “Radius $r = 0.01$ ”, “Radius $r = 0.10$ ” and “Radius $r = 0.30$ ”, the set of test points is 1,000 draws of state variables located on spheres with radii 0.01, 0.10 and 0.30, respectively; for panel “stochastic simulation”, the set of test points is a stochastic simulation of 10,000 periods. An entry “-” is used if the accuracy measure is below -10 (such errors are viewed as negligible).

Model equations are as follows: “EulerEq” is Euler equation (15); “MUCons” equates the (scaled) marginal utility of consumption to the Lagrange multiplier, see (13); “MULabor” equates the (scaled) marginal utility of labor to marginal productivity of labor multiplied by the Lagrange multiplier, see (14); “WorResConst” is world resource constraint (10); “Overall” is the maximum error across all the model’s equations; “Solution time” is time for computing a solution, and “Time to run test” is time for running a test in seconds; and “CGA” and “SMOL” are abbreviations for the cluster-grid, and Smolyak’s methods, respectively.

Table 2: Multi-country model with $N = 8$ countries: maximum absolute errors by equation on a sphere and on a stochastic simulation.

	1st order		2nd order		3rd order		Accuracy frontier in Kollmann et al. (2011b)
	PER	HYB	PER	HYB	PER	HYB	
Solution time	0.03	0.03	0.06	0.06	1.32	1.32	
Radius $r = 0.01$							
EulerEq	-6.26	-3.85	-7.07	-6.36	-7.85	-7.37	
MUCons	-5.28	-	-5.41	-	-6.23	-	
MULabor	-4.60	-	-5.34	-	-6.57	-	
WorResConst	-5.25	-	-6.22	-	-7.65	-	
Overall	-4.60	-3.85	-5.34	-6.36	-6.23	-7.37	-6.03 (CGA)
Time to run test	0.86	14.75	3.29	13.77	26.73	36.19	
Radius $r = 0.10$							
EulerEq	-4.20	-3.23	-5.04	-4.63	-6.76	-6.11	
MUCons	-3.28	-	-4.54	-	-6.02	-	
MULabor	-2.55	-	-3.93	-	-5.48	-	
WorResConst	-3.25	-	-4.75	-	-6.18	-	
Overall	-2.55	-3.23	-3.93	-4.63	-5.48	-6.11	-4.96 (CGA)
Time to run test	0.85	15.37	3.29	15.19	27.00	36.59	
Radius $r = 0.30$							
EulerEq	-3.17	-2.14	-3.57	-3.06	-4.68	-4.22	
MUCons	-2.32	-	-3.77	-	-4.74	-	
MULabor	-1.47	-	-2.46	-	-3.51	-	
WorResConst	-2.29	-	-3.38	-	-4.29	-	
Overall	-1.47	-2.14	-2.46	-3.06	-3.51	-4.22	-4.07 (SMOL)
Time to run test	0.85	17.37	3.36	17.93	26.97	39.46	
Stochastic simulation							
EulerEq	-3.53	-2.30	-3.49	-3.45	-4.37	-5.02	
MUCons	-1.79	-	-2.66	-	-3.52	-	
MULabor	-1.58	-	-2.40	-	-3.22	-	
ResConst	-2.22	-	-3.44	-	-4.48	-	
Overall	-1.58	-2.30	-2.40	-3.45	-3.22	-5.02	-4.54 (CGA)
Time to run test	8.48	157.77	34.48	160.56	269.86	373.92	

Notes: For each model equation (listed in the 1st column), the table reports maximum absolute errors in \log_{10} units across countries and test points. For panels “Radius $r = 0.01$ ”, “Radius $r = 0.10$ ” and “Radius $r = 0.30$ ”, the set of test points is 1,000 draws of state variables located on spheres with radii 0.01, 0.10 and 0.30, respectively; for panel “stochastic simulation”, the set of test points is a stochastic simulation of 10,000 periods. An entry “-” is used if the accuracy measure is below -10 (such errors are viewed as negligible).

Model equations are as follows: “EulerEq” is Euler equation (15); “MUCons” equates the (scaled) marginal utility of consumption to the Lagrange multiplier, see (13); “MULabor” equates the (scaled) marginal utility of labor to marginal productivity of labor multiplied by the Lagrange multiplier, see (14); “WorResConst” is world resource constraint (10); “Overall” is the maximum error across all the model’s equations; “Solution time” is time for computing a solution, and “Time to run test” is time for running a test in seconds; and “CGA” and “SMOL” are abbreviations for the cluster-grid, and Smolyak’s methods, respectively.

Table 3: Multi-country model with $N = 2$ countries: maximum differences across simulated series.

Variable	1st order		2nd order		3rd order	
	PER	HYB	PER	HYB	PER	HYB
Consumption of country 1						
1st order HYB	-2.40					
2nd order PER	-2.54	-2.63				
2nd order HYB	-2.53	-2.66	-3.44			
3rd order PER	-2.54	-2.65	-3.42	-3.91		
3rd order HYB	-2.54	-2.65	-3.46	-3.91	-4.44	
CGA	-2.54	-2.64	-3.45	-4.02	-4.21	-4.17
Consumption of country 2						
1st order HYB	-1.74					
2nd order PER	-1.88	-2.04				
2nd order HYB	-1.88	-2.06	-3.09			
3rd order PER	-1.89	-2.05	-3.32	-3.31		
3rd order HYB	-1.89	-2.05	-3.35	-3.31	-4.34	
CGA	-1.88	-2.04	-3.19	-3.42	-3.65	-3.57
Labor supply of country 1						
1st order HYB	-2.75					
2nd order PER	-3.09	-3.01				
2nd order HYB	-3.08	-3.02	-3.58			
3rd order PER	-3.07	-3.02	-3.57	-4.36		
3rd order HYB	-3.07	-3.02	-3.62	-4.36	-4.53	
CGA	-3.08	-3.02	-3.62	-4.46	-4.48	-4.66
Labor supply of country 2						
1st order HYB	-2.15					
2nd order PER	-2.28	-2.16				
2nd order HYB	-2.27	-2.17	-2.95			
3rd order PER	-2.28	-2.16	-2.88	-3.48		
3rd order HYB	-2.28	-2.16	-2.94	-3.44	-3.71	
CGA	-2.28	-2.15	-2.94	-3.53	-3.65	-3.73
Investment of country 1						
1st order HYB	-1.39					
2nd order PER	-1.42	-1.41				
2nd order HYB	-1.42	-1.42	-2.85			
3rd order PER	-1.41	-1.42	-2.70	-2.73		
3rd order HYB	-1.41	-1.42	-2.70	-2.73	-4.12	
CGA	-1.41	-1.42	-2.69	-2.84	-2.84	-2.85
Investment of country 2						
1st order HYB	-1.33					
2nd order PER	-1.23	-1.50				
2nd order HYB	-1.23	-1.51	-2.87			
3rd order PER	-1.24	-1.51	-2.66	-2.69		
3rd order HYB	-1.24	-1.51	-2.67	-2.69	-4.17	
CGA	-1.23	-1.50	-2.71	-2.80	-2.84	-2.85
Capital of country 1						
1st order HYB	-1.94					
2nd order PER	-1.82	-1.97				
2nd order HYB	-1.81	-1.99	-3.20			
3rd order PER	-1.81	-1.99	-3.18	-3.17		
3rd order HYB	-1.81	-1.99	-3.20	-3.16	-4.51	
CGA	-1.81	-1.99	-3.28	-3.48	-3.45	-3.44
Capital of country 2						
1st order HYB	-1.95					
2nd order PER	-1.81	-2.19				
2nd order HYB	-1.82	-2.22	-3.31			
3rd order PER	-1.82	-2.20	-3.23	-3.54		
3rd order HYB	-1.82	-2.21	-3.24	-3.55	-4.62	
CGA	-1.82	-2.22	-3.30	-3.69	-3.51	-3.53

Notes: For each model variable, the table reports maximum absolute values (in \log_{10} units) of relative differences between simulated series. The comparison is done pairwise across solution methods. “CGA” is an abbreviation for the cluster-grid method.