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Getting Normalization Right: Dealing with 'Dimensional Constants' in Macroeconomics [*]

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Abstract

We contribute to a recent literature on the normalization, calibration and estimation of CES production functions. The problem arises because CES 'share' parameters are not in fact shares, but depend on underlying dimensions - they are 'dimensional constants' in other words. It follows that such parameters cannot be calibrated, nor estimated unless the choice of units is made explicit. We use an RBC model to demonstrate two equivalent solutions. The standard one expresses the production function in deviation form about some reference point, usually the steady state of the model. Our alternative, 're-parametrization', expresses dimensional constants in terms of a new dimensionless (share) parameter and all remaining dimensionless ones. We show that our 're-parametrization' method is equivalent and arguably more straightforward than the standard normalization in deviation form. We then examine a similar problem of dimensional constants for CES utility functions in a two-sector model and in a small open economy model; then re-parametrization is the only solution to the problem, showing that our approach is in fact more general.

JEL Classification: E23, E32, E37
Keywords: CES production function, normalization, CES utility function.

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1 Introduction

The concept of the normalization and calibration of CES production functions is at the centre of a rapidly increasing literature in macroeconomics. In this paper, we attempt to clarify the issue. We propose an equivalent way of resolving the problem of normalization to that used in the literature that we call 're-parametrization'. Our paper applies dimensional analysis, usually neglected in economics, showing that normalization is not just a technical procedure and it is not specific to CES production functions. Indeed we show that normalization is needed more generally when dealing with 'dimensional constants' and it is usually done implicitly, if not explicitly.

The CES production function is not a novel concept and has been used extensively in many areas of economics since the middle of the previous century. The CES production function appears[1] in Solow (1956) Nobel Prize-winning essay and it has been subsequently generalized by Arrow *et al.* (1961).

A few years later De Jong (1967) and De Jong and Kumar (1972) pointed out, by applying 'dimensional analysis', that sometimes economists use forms of production functions that lack the crucial property of 'dimensional homogeneity' – i.e., both sides of the equation must have the same dimensions. This is not a problem for the usual Cobb-Douglas or Leontief specification, but one must be careful in considering dimensions when formulating a CES production function. Indeed they show that the CES functions as specified in Solow (1956) and Arrow *et al.* (1961) were not dimensionally homogeneous.[2]

Possibly because of this dimensionality issue and/or also for reasons of better analytical tractability, subsequent works in business cycle macroeconomics extensively used the Cobb-Douglas production function, almost forgetting the CES function defined by Arrow *et al.* (1961). Another reason might be the empirical observation that factor shares have been approximately constant over time. Indeed this observation has been the major justi-

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[1] Here we confine ourselves to the use of CES functions applied to Macroeconomics. A full review is beyond the scope of this paper. For a full review of the literature please refer to Klump *et al.* (2011). See La Grandville (2009) for a general discussion on CES production functions.

[2] See De Jong (1967) at pages 38-46 for the details and the discussion of why the production function as defined in Solow (1956) at page 77 and in Arrow *et al.* (1961) at page 230 are not dimensionally homogeneous because the technical parameters are treated as pure numbers whereas they *are elements of different dimensions and therefore are not additive in a conceptually meaningful way.*

fication of the use of Cobb-Douglas production function in the RBC literature.[3] However while the assumption of constant factor shares might be reasonable in growth models, it has been shown that (see for example Blanchard (1997), Jones (2003, 2005), McAdam and Willman (2011) and Ríos Rull and Santeulália-Llopis (2010)) such shares do fluctuate at business cycle frequencies. Anyway business cycle models have largely disregarded this issue and still today maintain the Cobb-Douglas hypothesis.

Recently there is an increasingly empirical evidence going in favour of CES production functions, in particular at business cycle frequencies, with elasticity of substitution well below unity (e.g., Klump *et al.* (2007), Chirinko (2008) and León-Ledesma *et al.* (2010)). The resurrection of the CES production function is due to La Grandville (1989) who extended the findings in Solow (1956) and introduced the concept of normalization. Normalization, i.e. expressing the production function in terms of index numbers, is important so that the parameters of the CES are deep (*dimensionless*) and not a mixture of production parameters which depends on the choice of units. La Grandville (1989) achieved this by normalizing the CES function at some chosen baseline values for the three following variables: the capital-labour ratio, per capita income and marginal rate of substitution. By doing so, one can avoid arbitrary results and express the efficiency and the distributional parameters of the CES as a function of the point of normalization and the elasticity of substitution.

The La Grandville procedure is needed when the researcher is interested in comparing economies which are distinguished only by their elasticity of substitution, as stressed by Klump and de La Grandville (2000). Indeed Klump and Preissler (2000) and Klump and de La Grandville (2000) explain that the normalized CES production function employed permits one to compare results with steady-state allocations and factor income shares that are constant as the elasticity of substitution is changed. The normalization procedure identifies then a family of CES production functions that are distinguished only by the elasticity parameter, and not by the steady-state allocations. This point is particularly important for business cycle models which use steady-state values about which to approximate the model's local dynamics up to the first order. In practice, normalization consists of recalibrating (or re-parameterizing) the model to match the data each time

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[3] See for example Cooley (1995) Chapter 1 page 16.

the elasticity parameter is varied. This is why Klump and Saam (2008) talk about arbitrary and inconsistent results if the CES function is not correctly normalized. Klump and de La Grandville (2000) also stressed that one of their objectives was to advocate the use of normalized CES functions in growth models.[4] A normalized CES production function approach has been used as well to investigate the implication of capital-labour substitution for equilibrium indeterminacy (Guo and Lansing (2009)). On the empirical side, León-Ledesma *et al.* (2010) show that normalization improves empirical identification.[5]

As recently stressed in Cantore *et al.* (2010), in a business cycle context, normalization (or re-parametrization) is also necessary so that we choose a normalization point corresponding to a steady state where factor shares directly map to certain CES parameters. Using non-normalized production functions not only obscures calibration results, but could also affect dynamic responses to shocks as the elasticity of output with respect to production inputs can change at different steady states. When you compare dynamic responses for different values of the elasticity of substitution a meaningful and consistent comparison requires analysing the models at the same normalization point. This is not necessary when working with Cobb-Douglas functions, since factor shares do not change, and hence it is not common practice in calibration of business cycle models.

Usually in this literature, normalization is presented as a technical procedure that applies only when dealing with the CES production function. Our aim here is to clarify the issue by using dimensional analysis to show that normalization always applies but is usually done implicitly (like in the Cobb-Douglas case, due to its multiplicative forms) and has to do with the presence of 'dimensional constants' which require a choice of units.

The second contribution of this paper regards the generalization of normalization beyond production theory. Although, to the best of our knowledge, the literature has so far confined the issue of normalization only for CES production functions, here we also relate the issue to CES utility functions in multi-sectoral models. We show that, while for the case of CES production functions the standard normalization approach and the

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[4] Furthermore they show how their approach is preferable to the one proposed in Barro and Sala-i-Martin (2004) (pp.68-74) which is also proven to be inconsistent in Klump and Preissler (2000).

[5] They show that using a normalized approach permits to overcome the 'impossibility theorem' stated by Diamond *et al.* (1978) and simultaneously identify the elasticity of substitution and biased technical change.

re-parametrization one presented here are equivalent, in the case of CES utility function in two-sector and open economy models re-parametrization is the only solution.

Given the focus of the paper on macroeconomic models of the business cycle, which are usually approximated up to the first order around the non stochastic steady state, the latter result is proved by showing that: (i) in the case of the one-sector RBC model the log-linearization can be expressed entirely in terms of dimensionless parameters; (ii) in the case of the two-sector (and open economy) model the linearization does depend on parameters that are not dimensionless (ie. depends on the choice of units).

The paper proceeds as follows. Section 2 introduces dimensional analysis, presents the normalization issue in a one-sector standard RBC model and sets out our two equivalent approaches. The standard one is to express the production function in deviation form about some reference point, usually the steady state of the model. The alternative we refer to as 're-parametrization' is conceptually more straightforward. This identifies parameters that are not dimensionless and in the absence of specifying dimensions cannot be quantified. It follows that such 'dimensional parameters' cannot be calibrated, nor estimated unless the choice of units is made explicit. Re-parametrization involves introducing a new share (and therefore) dimensionless parameter and expressing dimensional parameter in terms of this parameter and all remaining dimensionless ones. Sections 3 and 4 discuss the normalization and re-parametrization of CES utility functions respectively in a two-sector RBC model and in an open economy model. Finally Section 5 concludes.

2 Dimensional Constants in the CES Production Function

One can think of 'normalization' as removing the problem that always arises from the fact that labour and capital are measured in different units, but the units of measurement are not specified. Under Cobb-Douglas, normalization is implicit since, due to its multiplicative form, differences in units are absorbed by a scaling assumption that relates the units, whatever they are, to each other. The CES function, by contrast, is non-linear in logs, and so, unless correctly normalized, out of its three key parameters - the efficiency parameter, the distribution parameter, the substitution elasticity - only the latter is dimensionless. The other two parameters turn out to be affected by the size of the substitution elasticity and factor income shares. Then if one is interested in model sensitivity with respect to production parameters, normalization in needed to avoid arbitrary comparisons and to make sure that inference based on impulse-response functions is correct and not driven by the choice of units.

2.1 The CES Production Function

We start with a general CES production function (and Cobb-Douglas as a special case) in dynamic form suitable for use in a DSGE model

$$\begin{aligned} Y_t &= \left[\alpha_k (ZK_t K_t)^{\psi} + \alpha_n (ZN_t N_t)^{\psi}\right]^{\frac{1}{\psi}} \ ; \psi \neq 0 \ \& \ \alpha_k + \alpha_n \neq 1 \\ &= (ZK_t K_t)^{\alpha_k} (ZN_t N_t)^{\alpha_n} \ ; \psi \to 0 \ \& \ \alpha_k + \alpha_n = 1 \end{aligned} \tag{1}$$

where Y_t, K_t, N_t are output, capital and labour inputs respectively at time t, ZN_t and ZK_t representing respectively labour-augmenting and capital augmenting technical change, and $\psi \in (-\infty, 1]$ is the substitution parameter and α_k and α_n are sometimes referred as distribution parameters. The way we present the production function in (1) goes back to Pitchford (1960) who generalized the one presented in Solow (1956). Also De Jong (1967) and De Jong and Kumar (1972) use a similar formulation, that does not restrict the two distribution parameters (α_k and α_n) to sum up to 1, in order to show the dimensionality problem in the formulation by Solow and Arrow *et al.* (1961). As discussed in Klump and Preissler (2000) and Klump *et al.* (2011) in the literature different ways of expressing the CES production function have been used[6] with the one proposed by Arrow *et al.* (1961),

$$Y_t = C\left[\alpha (ZK_t K_t)^{\psi} + (1-\alpha)(ZN_t N_t)^{\psi}\right]^{\frac{1}{\psi}} \tag{2}$$

probably being the most used. However it is straightforward to show that the formulation in (2) is equivalent to the one presented in (1) with $C = (\alpha_k + \alpha_n)^{\psi}$ and $\alpha = \frac{\alpha_k}{\alpha_k + \alpha_n}$.

Calling $\sigma \equiv \frac{1}{1-\psi}$ the elasticity of substitution between capital and labour,[7] then with

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[6] Klump and Preissler (2000) also discuss why the formulation used by Barro and Sala-i-Martin (2004) is inconsistent.

[7] The elasticity of substitution for the case of perfect competition, where all the product is used to remunerate factor of productions, is defined as the elasticity of the capital/labour ratio with respect to the wage/capital rental ratio. Then calling W the wage and $R + \delta$ the rental rate of capital we can define the elasticity as follows:

$$\sigma = \frac{d\frac{K}{N}\frac{N}{K}}{d\frac{W}{R+\delta}\frac{R+\delta}{W}}.$$

See La Grandville (2009) for a more detailed discussion.

$\psi \in (-\infty, 1)$, $\sigma \in (0, +\infty)$. When $\sigma = 0 \Rightarrow \psi = -\infty$ we have the Leontief case, $\sigma = 1 \Rightarrow \psi = 0$ collapses to the usual Cobb-Douglas case and as $\sigma \to \infty \Rightarrow \psi \to 1$ capital and labour become perfect substitutes.

2.2 Dimensional Analysis

From the outset a discussion of dimensions and dimensional analysis is essential. ZK and ZN are not measures of efficiency as they depend on the units of output and inputs (i.e., are not dimensionless and the problem of normalization arises because unless $\psi \to 0$, α in (2) is not a share and in fact is also not dimensionless.

As in every science a choice of primary dimensions must be made. It is useful to consider the example of classical mechanics and Newton's famous law of gravitation. This states that two masses m_1 and m_2 a distance r apart attracts each other with a force given by

$$F = G\frac{m_1 m_2}{r^2} \tag{3}$$

where G is a constant. The important point is that the value of G depends on the choice of units. Dimensional analysis essentially imposes *dimensional homogeneity* on a relationship, that is the requirement that both sides of an equation must have the same dimensions to be meaningful. Let $[F]$ denote the dimension of force and define $[m]$ and $[r]$ similarly. It follows that the dimension of the constant G is $[Fr^2m^{-2}]$ and in fact it turns out that in metric units $G = 6.670 \times 10^{-11}$ newton×(meters)^2 per (kilogram)^2. Here a newton is a unit of force and is a secondary dimension. In terms of primary dimensions of mass, distance and time, by Newton's second law force equals mass × acceleration and hence $F \in [mrT^{-2}]$ where T is time so in terms of primary dimensions $G \in [r^3T^{-2}m^{-1}]$ and has metric units of cubic meters per (kilogram×second^2).

In macroeconomics we do not distinguish between the millions of goods produced nor between the many types of labour. Instead we construct composite measures of output, labour and capital which have dimensions, but often the unit of measurement is not made explicit. Let our primary dimensions of such composites be output (R_y), capital (R_k), labour (R_l) and time (T). Then the flow output per period has dimensions R_yT^{-1}, labour per period has dimensions R_lT^{-1}, whilst capital is a stock of accumulated output with dimension $R_k = R_yT^{-1}T = R_y$. Consider first the Cobb-Douglas production function in a no-growth steady state $Y = (ZKK)^{\alpha}(ZNN)^{1-\alpha}$ where we define $\alpha_k = \alpha$. Then by dimensional homogeneity

$$Y \in [R_y T^{-1}] = (ZKK)^{\alpha}(ZNN)^{1-\alpha} \in [ZK^{\alpha} ZN^{1-\alpha} R_y^{\alpha}(R_l T^{-1})^{1-\alpha}] \tag{4}$$

It follows that the composite constant, $ZK^{\alpha}ZN^{1-\alpha} \in [R_y^{1-\alpha}R_l^{\alpha-1}T^{-\alpha}]$. For example, if labour is the only input, $\alpha = 0$, ZK disappears and $ZN \in R_y R_l^{-1}$; that is output per unit of labour or labour productivity. If capital is the only input, $\alpha = 1$, ZN disappears and $ZK \in T^{-1}$ which enables a stock to be related to a flow.[8]

These steady-state 'efficiency parameters' ZK and ZN are then constants that depend on the choice of units. They are in other words *dimensional constants.* De Jong (1967) proposes the following procedure that avoids specifying units of measurement. Define a steady-state baseline point (later the literature refers to this as a 'normalization' point) $Y = Y_0$, $K = K_0$ and $N = N_0$ that satisfy the Cobb-Douglas production function. Then $Y_0 = (ZKK_0)^{\alpha}(ZNN_0)^{1-\alpha}$. Dividing the two forms of the production function we have what De Jong refers to as the "revised version"

$$\frac{Y}{Y_0} = \left(\frac{K}{K_0}\right)^{\alpha}\left(\frac{N}{N_0}\right)^{1-\alpha} \tag{5}$$

Now all the ratios $\frac{Y}{Y_0}$ etc are dimensionless and the troublesome dimensional constants ZN and ZK have been eliminated. In fact there is a simpler way of handling these dimensional constants. Units can be chosen so that when $N = 1$ and $K = 1$, then $Y = 1$ implying $ZK^{\alpha}ZN^{1-\alpha} = 1$. We do not need to specify units of measurement to do this. What we are saying is that whatever our units of say output and labour are, we can define the units of capital and time so that $Y = N = K = 1$ and hence $ZK^{\alpha}ZN^{1-\alpha} = 1$.[9] Since we wish the ZK and ZN to be independent of α it follows that $ZK = ZN = 1$.

Things are not so straightforward when we generalize to a CES production function. As before we can put $ZK = ZN = 1$ and dispose of one problem of dimensional constants.

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[8] For some, these dimensional requirements pose a fundamental problem with the notion of a production function - see Barnett (2004)

[9] For example suppose 1 kilogram of steel (capital) combines with 4 hours of labour to give one unit of a product per day. In fact later we will define N to be a proportion of a day and therefore dimensionless; so $N = 1$ combines with 2 kilograms of steel to give 2 units of the product. Then redefine a unit of output to be the latter and a unit of capital to be 2 kilograms of steel.

Then applying dimensional homogeneity we have that

$$\begin{aligned}\alpha_n &\in [R_y^{\psi} R_l^{-\psi}] \quad (6)\\ \alpha_k &\in [T^{-\psi}] \quad (7)\end{aligned}$$

so the dimensions of both parameters depend on the distribution parameter ψ. In addition, α_n is a dimensional constant depending on units of output and labour whereas α_k only depends on the unit of time which we *do* specify in our macroeconomic models and data. Equivalently C and α in (2) are also dimensional constants. It follows that their values will change simply by a different choice of units and the actual data will not be able to pin down their values.[10]

2.3 RBC Model

We embed the CES production function in a very standard RBC model with no costs of investment as in Cantore *et al.* (2010). It consists of a household's utility function, the first-order conditions for intertemporal savings and consumption, C_t (the Euler Equation), their labour supply decisions, a CES production function for firms, their first-order conditions for labour and capital inputs and an output equilibrium.

$$\begin{aligned}\text{Utility} : \Lambda_t &= \Lambda(C_t, 1 - N_t) && (8)\\ \text{Euler} : \Lambda_{C,t} &= \beta E_t\left[(1 + R_{t+1})\Lambda_{C,t+1}\right] && (9)\\ \text{Labour Supply} : W_t &= -\frac{\Lambda_{N,t}}{\Lambda_{C,t}} && (10)\\ \text{Production Function} : Y_t &= F(ZK_t, ZN_t, K_t, N_t) && (11)\\ \text{Labour Demand} : F_{N,t} \equiv MPL_t &= W_t && (12)\\ \text{Capital Demand} : F_{K,t} \equiv MPK_t &= R_t + \delta && (13)\\ \text{Equilibrium} : Y_t &= C_t + K_{t+1} - (1 - \delta)K_t + G_t && (14)\end{aligned}$$

where $\Lambda_{L,t}$ and $\Lambda_{C,t}$ are respectively the marginal utilities of labour supply and consumption, and G_t is government spending. Seven equations (8)–(14) describe and equilibrium in seven real variables $\{\Lambda_t\}$, $\{C_t\}$, $\{N_t\}$, $\{R_t\}$, $\{W_t\}$, $\{Y_t\}$ and $\{K_t\}$ given exogenous pro-

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[10] This is neatly demonstrated in Klump *et al.* (2011) in an example where different values of C and α can be generated simply by changing the units of capital - see their Table 3.

cesses for $\{ZK_t\}$, $\{ZN_t\}$ and $\{G_t\}$ and the initial value of the one predetermined variable in the model, beginning of period capital stock, K_t.

We next choose functional forms. The production function is CES as above and the utility function is standard and chosen to be compatible with a balanced-growth path:

$$\Lambda_t = \frac{(C_t^{(1-\varrho)}(1-N_t)^{\varrho})^{1-\sigma_c}-1}{1-\sigma_c} \tag{15}$$

Marginal utilities and marginal products (the latter equated with factor prices) are now given by

$$\begin{aligned}
\Lambda_{C,t} &= (1-\varrho)C_t^{(1-\varrho)(1-\sigma_c)-1}(1-N_t)^{\varrho(1-\sigma_c)} && (16)\\
\Lambda_{N,t} &= \varrho C_t^{(1-\varrho)(1-\sigma_c)}(1-N_t)^{\varrho(1-\sigma_c)-1} && (17)\\
F_{N,t} &= \frac{Y_t}{N_t}\left[\frac{\alpha_n(ZN_tN_t)^{\psi}}{\alpha_k(ZK_tK_t)^{\psi}+\alpha_n(ZN_tN_t)^{\psi}}\right] = \alpha_n ZN_t^{\psi}\left(\frac{Y_t}{N_t}\right)^{1-\psi} = W_t && (18)\\
F_{K,t} &= \frac{Y_t}{K_t}\left[\frac{\alpha_k(ZK_tK_t)^{\psi}}{\alpha_k(ZK_tK_t)^{\psi}+\alpha_n(ZN_tN_t)^{\psi}}\right] = \alpha_k ZK_t^{\psi}\left(\frac{Y_t}{K_t}\right)^{1-\psi} = R_t+\delta && (19)
\end{aligned}$$

and along with specified forms of the exogenous processes, this completes the specification of the model. The equilibrium of real variables depends on parameters ϱ, σ_c, δ, ψ, α_k and α_n. Of these ϱ, ψ and σ_c are dimensionless, δ depends on the unit of time, but unless $\psi = 0$ and the technology is Cobb-Douglas, α_k and α_n depend on the units chosen for factor inputs, namely machine units per period and labour units per period. To see this rewrite (18) and (19) in terms of factor shares

$$\begin{aligned}
\frac{W_tN_t}{Y_t} &= \alpha_n ZN_t^{\psi}\left(\frac{Y_t}{N_t}\right)^{-\psi} && (20)\\
\frac{(R_t+\delta)K_t}{Y_t} &= \alpha_k ZK_t^{\psi}\left(\frac{Y_t}{K_t}\right)^{-\psi} && (21)
\end{aligned}$$

from which

$$\frac{W_tN_t}{(R_t+\delta)} = \frac{\alpha_n}{\alpha_k}\left(\frac{ZK_tK_t}{ZN_tN_t}\right)^{-\psi} \tag{22}$$

Thus α_n (α_k) can be interpreted as the share of labour (capital) iff $\psi = 0$ and the production function is Cobb-Douglas. Otherwise the dimensions of α_k and α_n depend on those for $\left(\frac{ZK_tK_t}{ZN_tN_t}\right)^{\psi}$ which could be for example, (machine hours per effective person hours)$^{\psi}$. In our aggregate production functions we choose to avoid specifying unit of capital, labour and output. It is impossible to interpret and therefore to calibrate or estimate these 'share' parameters.

There are two ways to resolve this problem; 're-parameterize' the dimensional parameters α_k and α_n so that they are expressed in terms of dimensionless parameters to be estimated or calibrated, or 'normalize' the production function in terms of deviations from a steady state. We consider these in turn.

2.4 Re-parametrization of α_n and α_k

First write the balanced growth steady state of consumption Euler equation as

$$\frac{\bar{\Lambda}_{C,t+1}}{\bar{\Lambda}_{C,t}} = \left[\frac{\bar{C}_{t+1}}{\bar{C}_t}\right]^{(1-\varrho)(1-\sigma_c)-1} = (1+g)^{((1-\varrho)(1-\sigma_c)-1)} = \beta(1+R) \tag{23}$$

On the balanced-growth path (bgp) consumption, output, investment, capital stock, the real wage and government spending are growing at a common growth rate g driven by exogenous labour-technical change $\overline{ZN}_{t+1} = (1+g)\overline{ZN}_t$, but labour input N is constant.[11] As it is well-known, a bgp requires either Cobb-Douglas technology or that technical change must be driven solely by the labour-augmenting variety (see, for example, Jones (2005)).[12] Then $ZK_t = ZK$ must also be constant along the bgp. It is convenient to stationarize the bgp by defining stationary variables such as $Y \equiv \frac{\bar{Y}_t}{ZK\overline{ZN}_t}$.[13] Then the stationarized bgp is given by

$$\begin{aligned}
Y &= \left[\alpha_k K^{\psi} + \alpha_n N^{\psi}\right]^{\frac{1}{\psi}} && (24)\\
\frac{\varrho C}{(1-\varrho)(1-N)} &= W && (25)\\
\frac{Y}{N}\left[\frac{\alpha_n N^{\psi}}{\alpha_k K^{\psi} + \alpha_n N^{\psi}}\right] &= W && (26)\\
\frac{Y}{K}\left[\frac{\alpha_k K^{\psi}}{\alpha_k K^{\psi} + \alpha_n N^{\psi}}\right] &= R + \delta && (27)\\
I &= (\delta + g)K && (28)\\
Y &= C + I + G && (29)
\end{aligned}$$

which together with (23) defines the bgp.

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[11] If output, consumption etc are defined in per capita terms then N can be considered as the proportion of the available time at work and is therefore both stationary and dimensionless.

[12] Recently León-Ledesma and Satchi (2010) have demonstrated that by using a slightly modified CES production function it is possible to introduce capital augmenting technical change as well along the bgp. Here for reasons of simplicity we do not consider that case.

[13] The full model can also be stationarized in the same way by dividing Y_t, C_t, etc by $ZK\overline{ZN}_t$.

We can now define

$$\pi \equiv \frac{\alpha_n N^{\psi}}{\alpha_k K^{\psi} + \alpha_n N^{\psi}} = \frac{WN}{(R+\delta)K + WN} \tag{30}$$

$$1 - \pi \equiv \frac{\alpha_k K^{\psi}}{\alpha_k K^{\psi} + \alpha_n N^{\psi}} = \frac{(R+\delta)K}{(R+\delta)K + WN} \tag{31}$$

which are the labour and capital share on the bgp and are both dimensionless and stationary. Then using (24), (30) and (31) we obtain our *re-parametrization* of α_n and α_k:

$$\alpha_n = \pi \left(\frac{Y}{N}\right)^{\psi} \tag{32}$$

$$\alpha_k = (1-\pi) \left(\frac{Y}{K}\right)^{\psi} \tag{33}$$

Note that $\alpha_n = \pi$ and $\alpha_k = 1 - \pi$ at $\psi = 0$, the Cobb-Douglas case.[14] Before proceeding we need to apply dimensional analysis. From (32) we see that $\alpha_n \in [R_y^{\psi} R_l^{-\psi}]$ confirming (6). From (58) and (31) we see that $\frac{Y}{K}(1-\pi) = R + \delta$. Hence we have that

$$\alpha_k = (1-\pi)^{1-\psi}(R+\delta)^{\psi} \in [T^{-\psi}] \tag{34}$$

confirming (7).

To complete the description of the model including the parameters α_k and α_n we need to characterize the bgp steady state. From (58) – (58) we have the *shares*

$$\frac{K(R+\delta)}{Y} = \frac{\bar{K}_t(R+\delta)}{\bar{Y}_t} = 1 - \pi \tag{35}$$

$$\frac{WN}{Y} = \frac{\bar{W}_t N}{\bar{Y}_t} = \pi \tag{36}$$

$$\frac{I}{Y} = \frac{\bar{I}_t}{\bar{Y}_t} = \frac{(\delta+g)K}{Y} = \frac{\pi(\delta+g)}{(R+\delta)} \tag{37}$$

$$\frac{C}{Y} = 1 - \frac{\bar{I}_t}{\bar{Y}_t} - \frac{\bar{G}_t}{\bar{Y}_t} \tag{38}$$

which are both dimensionless and independent of the production elasticity ψ, as is the real interest rate. Using (25) we have

$$\frac{(1-N)(1-\varrho)}{N\varrho}\frac{\frac{WN}{Y}}{\frac{C}{Y}} = \frac{(1-N)(1-\varrho)\pi}{N\varrho} \tag{39}$$

from which N is obtained. The steady state consumption Euler equation (23) determines R and hence $\frac{K}{Y}$ from (35). To recover *levels* along the bgp first put $\bar{Y}_t = ZK_t\overline{ZN}_tY$

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[14] And as argued before if $\pi \in (0,1)$ $\alpha_k + \alpha_n = 1$ iff $\psi = 0$.

etc. There is one more dimensional issue: the specification of dimensional constants $\overline{ZN}_0$ and ZK. As argued earlier by choice of units in the steady state at $t = 0$ we can put $\overline{ZN}_0 = ZK = 1$. This completes the bgp steady-state equilibrium which is now defined only in terms of *dimensionless* parameters ϱ, σ_c, ψ, π and δ which depends on the unit of time. In (32) and (33) dimensional parameters expressed in terms of other endogenous variables Y, N and K are now themselves functions of $\theta \equiv [\sigma, \psi, \pi, \delta]$. Therefore $\alpha_n = \alpha_n(\theta)$, and $\alpha_k = \alpha_k(\theta)$ which expresses why we refer to this procedure as re-parametrization.

To calibrate these dimensionless parameters and δ, if we have data for R, g, π, $\frac{C}{Y}$, $\frac{I}{Y}$ and N we can pin down δ and ϱ from (37) and (39) respectively. Then (23) can be used to calibrate one out of the two remaining parameters β and σ_c. Since there is a sizeable literature on the microeconometric estimation of the latter risk-aversion parameter, it is usual to use this and calibrate β.

2.5 The Production Function in Deviation Form

This simply bypasses the need to retain α_k and α_n and writes the dynamic production function in deviation form about its steady state as

$$\frac{Y_t}{\bar{Y}_t} = \left[\frac{\alpha_k ZK_t K_t^{\psi} + \alpha_n (ZN_t N_t)^{\psi}}{\alpha_k \bar{K}_t^{\psi} + \alpha_n (\overline{ZN}_t N)^{\psi}}\right]^{\frac{1}{\psi}} = \left[\frac{\alpha_k \left(\frac{ZK_t K_t}{\bar{K}_t}\right)^{\psi}}{\alpha_k + \alpha_n \left(\frac{\overline{ZN}_t N}{\bar{K}_t}\right)^{\psi}} + \frac{\alpha_n \left(\frac{ZN_t N_t}{\overline{ZN}_t N_t}\right)^{\psi}}{\alpha_k \left(\frac{\bar{K}_t}{\overline{ZN}_t N}\right)^{\psi} + \alpha_n}\right]^{\frac{1}{\psi}}$$

From the steady-state of the first order conditions, and from (30) in particular, we can write this simply as

$$\frac{Y_t}{\bar{Y}_t} = \left[(1-\pi)\left(\frac{ZK_t K_t}{\bar{K}_t}\right)^{\psi} + \pi\left(\frac{ZN_t N_t}{\overline{ZN}_t N}\right)^{\psi}\right]^{\frac{1}{\psi}} \tag{40}$$

as in Cantore *et al.* (2010). The steady-state is characterized as before and again involves a further 'normalization' $\bar{Y}_0 = \overline{ZN}_0 = ZK = 1$.[15]

Re-parametrization and writing the production function in deviation form are two equivalent ways of eliminating the dimensional parameters in the CES production. However following Arrow *et al.* (1961) it is possible to estimate a non-normalized CES production function of the form (2). Using aggregate private non-farm output in the US (1929

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[15] Which is almost identical to the one used in Cantore *et al.* (2010) although they normalize as well hours worked to 1 using the accounting identity $\bar{Y} = (\bar{R} + \delta)\bar{K} + \bar{W}\bar{N}$.

- 49) they obtained $C = 0.584(1.0183)^t$, $\alpha = 0.481$ and $\psi = -0.756$. What can we make of these estimates? They are perfectly valid provided the units are made explicit. For this exercise these units for labour are person-years, for output \$m at 1939 prices per year and for capital the stock measured in \$m at 1939 prices. However as demonstrated in León-Ledesma *et al.* (2010) using Monte Carlo experiments there are enormous advantages in estimating a normalized CES production function arising from the fact that the parameters to be estimated are dimensionless; in particular the share parameter has a natural prior, a feature that is particularly pertinent in the estimation of DSGE models where Bayesian estimation is now standard. This econometric advantage of using a normalized form is in addition to a second advantage, alluded to in the introduction, that non-normalized production functions cannot be used to carry out comparative static exercise as the elasticity σ changes. Our dimensional analysis results (32) and (33) clearly show that the units of measurement must depend on ψ and therefore σ, so as we change the elasticity then the unit of measurement changes. Estimates on one choice of units cease to be valid as these change thus invalidating the comparison based on the original estimates of (32) and (33). By contrast, in our re-parametrization (32) and (33) become endogenous for a given dimensionless share parameter π.

2.6 Dynamic Set-up for Simulations

The two ways of addressing the normalization issue result in two equivalent set-ups for use in modelling software such as **Dynare**. These involve the following steps:

1. Solve for the bgp steady state for dimensionless variables consisting of consumption-output, capital-output, etc shares, labour supply (the proportion of hours worked) and the real interest rate. For the RBC model these are given by (23) and (35) – (38) and can be solved analytically in a sequential fashion. In general a numeral algorithm is required.

2. Choose units in a convenient way at $t = 0$. For our model we do so in such a way that along the bgp $\bar{Y}_0 = \overline{ZN}_0 = ZK = 1$.

3. Having solved for this bgp steady state, the model dynamics along the bgp in levels

is now obtained from the dynamics of output

$$\bar{Y}_t = ZK\overline{ZN}_t Y \tag{41}$$

$$\overline{ZN}_{t+1} = (1+g)\overline{ZN}_t \tag{42}$$

4. For the re-parametrization approach define output by the basic CES production function (1) with dimensional parameters α_n and α_k given by (32) and (33).

5. Or for what has now become the standard normalization approach set the production in deviation form (40).

6. Both forms follow from the same first-order conditions and are equivalent.

Figures 1 illustrate our re-parametrization as the elasticity of substitution parameter σ varies.[16] In our simulations consumption, investment and government spending ratios are constant as well as hours and real wages. What are changing are the two parameters α_k and α_n that are not dimensionless maintaining the same steady state for across σ (figure 2). *If we compute impulse response functions at any point along these graphs we would not get arbitrary results, in line with Klump and Saam (2008) and Cantore* et al. *(2010).*

As it is clear from the previous figures the parameters α_k and α_n are not dimensionless and for this reason when the model is set up with our first re-parametrization we cannot calibrate nor estimate the α's without specifying explicitly the choice of units; only the dimensionless parameter π can be meaningfully quantified.[17]

2.7 Linearization

Define lower case variables $x_t = \log \frac{X_t}{X}$ where X is the bgp stationarized steady state value of a trended variable. For the variable $r_t \equiv \log\left(\frac{1+R_t}{1+R}\right)$ is the log-linear *gross* real interest rate. Then using the 're-parametrization' approach presented in section 2.4 and substituting (33) and (32) it can be shown[18] that the log-linearized RBC model about the

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[16] Parameter values are $\pi = 0.6$, $g = 0$, $\sigma_c = 2.0$, $\beta = 0.99$, $\varrho = 0.6030$, $g_y \equiv \frac{G}{Y} = 0.2$ and $\delta = 0.025$

[17] The Dynare and Matlab programs are available from the authors on request.

[18] See Appendix A for the log-linearization of the production function.

BGP steady state takes the state-space form

$$
\begin{aligned}
k_{t+1} &= \frac{1-\delta}{1+g}k_t + \frac{\delta+g}{1+g}i_t \\
E_t[\lambda_{C,t+1}] &= \lambda_{C,t} - E_t[r_{t+1}] \\
\lambda_{C,t} &= -(1+(\sigma_c-1)(1-\varrho))c_t + (\sigma_c-1)\varrho\frac{N}{1-N}n_t \\
\lambda_{N,t} &= \lambda_{C,t} + c_t + \frac{N}{1-N}n_t \\
w_t &= \lambda_{N,t} - \lambda_{C,t} = c_t + \frac{N}{1-N}n_t \\
y_t &= \pi(n_t + a_t) + (1-\pi)k_t \\
y_t &= \frac{C}{Y}c_t + \frac{I}{Y}i_t + \frac{G}{Y}g_t \\
\frac{1+R}{R+\delta}r_t &= (1-\psi)(y_t - k_t) \\
w_t &= (1-\psi)(y_t - n_t) + \psi a_t
\end{aligned}
$$

From equation (40) it is clear that by using the production function in deviation form showed in section 2.5 we arrive at the same result. The importance of this result is that the log-linearization can be expressed entirely in terms of dimensionless parameters ϱ, σ_c, ψ and π and δ which depends on the unit of time, *and is no longer a function of* α_n *and* α_k. It follows that, *once the model is re-parameterized* the first-order dynamics and impulse response functions in the region of the steady-state are independent of these dimensional parameters.

2.8 Summary

The CES function is defined in terms of a 'share parameters' α_n and α_k which are not dimensionless (and therefore not shares) and consequently cannot be quantified without specifying the precise units of factors and output. We avoid doing this in macroeconomics so we need to either re-parameterize the model by expressing these dimensionless parameters in terms of the other endogenous variables and a newly introduced parameter, the long-run labour share of output, π or we need to express the production function in deviation form. If we include π in the vector of parameters θ, we can then express $\alpha_n = \alpha_n(\theta)$ and $\alpha_k = \alpha_k(\theta)$ in effect treating these 'parameters' as variables. Alternatively we can eliminate α_n and α_k altogether and formulate the production function in normalized form as a function of π. The log-linearized form of the model can be expressed in terms of π

and does not involve the dimensional parameters. Both set-ups require some further normalization (choice of units) for the steady states, but the model dynamics is independent of this choice.

3 Re-parametrization in Utility Functions: Two-Sector Model

We now show that a similar problem and solution arise with the parametrization of utility functions. Consider a 2-sector version of our one-sector RBC model. Factors of production are perfectly mobile so that factor prices are equalized. A proportion n_1 of household members supply labour hours $h_{1,t}$ in sector 1, a proportion $1 - n_1$ supply labour $h_{2,t}$ in sector 2. The two sectors produce goods that are imperfect substitutes with prices $P_{i,t}$, $i = 1, 2$. Quantities $C_{i,t}$, $I_{i,t}$, $G_{i,t}$ and $K_{i,t}$ are defined similarly. We assume each sector accumulates capital out of its own output. To simplify matters we confine ourselves to the case of only labour-augmenting change and put $ZK_{i,t} = 1$ and $ZN_{i,t} = A_{i,t}$.

First we construct a Dixit-Stiglitz CES consumption index and a corresponding price aggregate

$$C_t = \left[\mathrm{w}^{1-\phi} C_{1,t}^{\phi} + (1-\mathrm{w})^{1-\phi} C_{2,t}^{\phi}\right]^{\frac{1}{\phi}} \ ; \ \phi \in [-\infty, 1] \ , \phi \neq 0 \tag{43}$$

$$= C_{1,t}^{\mathrm{w}} C_{2,t}^{1-\mathrm{w}} \ ; \quad \phi = 0 \tag{44}$$

$$P_t = \left[\mathrm{w}(P_{1,t})^{1-\mu} + (1-\mathrm{w})(P_{2,t})^{1-\mu}\right]^{\frac{1}{1-\mu}} \ ; \ \mu \neq 1 \tag{45}$$

$$= P_{1,t}^{\mathrm{w}} P_{2,t}^{1-\mathrm{w}} \ ; \ \mu = 1 \tag{46}$$

where $\mu \equiv \frac{1}{1-\phi} \in [0, \infty]$. Then standard inter-temporal and intra-temporal decisions give

$$\text{Utility} : \Lambda_t = \Lambda(C_t, n_{1,t}, h_{1,t}, h_{2,t}) = n_{1,t} U(C_t, h_{1,t}) + (1 - n_{1,t}) U(C_t, h_{2,t})) \tag{47}$$

$$\text{Euler} : \Lambda_{C,t} = \beta E_t \left[(1 + R_{t+1}) \Lambda_{C,t+1}\right] \tag{48}$$

$$\text{FOC} \quad C_{1,t} : C_{1,t} = \mathrm{w} \left(\frac{P_{1,t}}{P_t}\right)^{-\mu} C_t \tag{49}$$

$$\text{FOC} \quad C_{2,t} : C_{2,t} = (1-\mathrm{w}) \left(\frac{P_{2,t}}{P_t}\right)^{-\mu} C_t \tag{50}$$

$$\text{Production Function}: Y_{i,t} = F(A_{i,t}, N_{i,t}, K_{i,t}) \tag{51}$$

$$N_{i,t} \equiv h_{i,t} n_{i,t} \tag{52}$$

$$\text{Labour Supply}: \frac{\Lambda_{h_i,t}}{\Lambda_{C,t}} = -W_t \tag{53}$$

$$\tag{54}$$

Note that (49), (50) and (45) imply that $P_{1,t}C_{1,t} + P_{1,t}C_{1,t} = P_tC_t$.

The firm's behaviour is summarized by:

$$\text{FOC} \quad N_{i,t}: \frac{P_{i,t}}{P_t} F_{N_i,t} = W_t \tag{55}$$

$$\text{FOC} \quad K_{i,t}: \frac{P_{i,t}}{P_t} F_{K_i,t} = R_t + \delta \tag{56}$$

The model is completed with an output equilibrium in each sector

$$Y_{i,t} = C_{i,t} + G_{i,t} + K_{i,t+1} - (1 - \delta_i)K_{i,t} \tag{57}$$

Functional form for $U(C_t, L_t)$ is chosen as for the one-sector model and $F(A_t, L_t, K_t)$ is assumed to be Cobb-Douglas, in order to focus on the utility function issue, but with different parameters in the two sectors. Equations (47) – (57) describe an equilibrium in Λ_t, C_t, $\frac{W_t}{P_t}$, $Y_{i,t}$, $h_{i,t}$, $\frac{P_{i,t}}{P_t}$, $K_{i,t}$, $I_{i,t}$, R_t given exogenous processes for $A_{i,t}$ and $G_{i,t}$ and parameters ϱ, σ_c, ψ_i, δ, $\alpha_{n,i}$, $\alpha_{k,i}$ and w for $i = 1, 2$. As before ϱ, ψ and σ_c are dimensionless, δ_i depends on the unit of time and $\alpha_{n,i}$, $\alpha_{k,i}$ depend on factor units (unless $\psi_i = 0$) and need to be replaced by dimensionless share parameters in the same fashion. But as we shall now see there is a further non-dimensionless parameter w, (unless $\phi = 0$). As for the one-sector model and the CES production function we explore two ways of dealing with this problem: re-parameterizing w or expressing the CES utility function in deviation form.

3.1 Re-parametrization of w

First we must set out the balanced-growth path (bgp) steady state. Defining stationary variables such as $C \equiv \bar{C}/\bar{A}_1$, the stationarized bgp is given by

$$\begin{aligned}
1+R &= (1+g)^{((1-\varrho)(1-\sigma_c)-1)} \\
Y_i &= N_i^{\pi_i} K_i^{1-\pi_i}\,;\ i=1,2 \\
N_1 &\equiv n_1 h_1 \\
N_2 &\equiv (1-n_1)\left(\frac{\bar{A}_{2,t} h_2}{\bar{A}_{1,t}}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\varrho C}{(1-\varrho)(1-h_1)\left(n_1+(1-n_1)\left(\frac{(1-h_1)}{(1-h_2)}\right)^{\varrho(\sigma_c-1)}\right)} &= W \\
\frac{\varrho C}{(1-\varrho)(1-h_2)\left(n_1\left(\frac{1-h_2}{1-h_1}\right)^{\varrho(\sigma_c-1)}+1-n_1\right)} &= W
\end{aligned}$$

$$\begin{aligned}
\frac{\pi_i Y_i}{N_i} &= W\,;\ i=1,2 \\
\frac{(1-\pi)Y_i}{K_i} &= R+\delta\,;\ i=1,2 \\
I_i &= (\delta+g)K_i\,;\ i=1,2 \\
Y_i &= C_i+I_i+G_i\,;\ i=1,2
\end{aligned}$$

$$C_1 = \mathrm{w}\left(\frac{P_1}{P}\right)^{-\mu} C \tag{58}$$

$$C_2 = (1-\mathrm{w})\left(\frac{P_2}{P}\right)^{-\mu} C \tag{59}$$

$$P^{1-\mu} = \mathrm{w}P_1^{1-\mu}+(1-\mathrm{w})P_2^{1-\mu} \tag{60}$$

$$G_1 = g_{y1} Y_1 \tag{61}$$

$$G_2 = g_{y2} Y_2 \tag{62}$$

From (58) we now have $\omega_c \equiv \frac{P_1 C_1}{PC} = \mathrm{w}\left(\frac{P_1}{P}\right)^{1-\mu}$ from which we can express w as

$$\mathrm{w} = \omega_C \left(\frac{P_1}{P}\right)^{\mu-1} = \mathrm{w}(\theta) \tag{63}$$

where $\theta \equiv [\mu,\ \sigma_c,\ \beta,\ g,\ \varrho,\ \delta_i,\ \pi_i,\ \psi_i\ \omega_C]$ which we refer to as *parametrization 1.* We can now see that w depends on the dimensionless share parameter ω_C and (unless $\mu = 1$)

on prices P_1 and P_2 on the bgp. The latter in turn depend on units of output so w is not dimensionless and therefore cannot be pinned down without an explicit choice of units of output.[19] However we can re-parameterize the model in terms of ω_C which is dimensionless and readily calibrated. Since $\left(\frac{P_1}{P}\right)$ can be expressed in terms of the dimensionless parameters μ, σ_c, β, g, ϱ, δ_i, π_i, ψ_i from (63) it follows that w can be expressed in terms of these plus our new parameter ω_C.

Parametrization 1 depends on the observation of the consumption share ω_C which might not always be available. In the case where the only available data are for the *output share* $\omega_Y \equiv \frac{P_1 Y_1}{PY}$, from (58) – (59) we have

$$\omega_Y \equiv \frac{P_1 Y_1}{PY} = \frac{\mathrm{w}\left(\frac{P_1}{P}\right)^{1-\mu} C + \left(\frac{P_1}{P}\right)(I_1 + G_1)}{\mathrm{w}\left(\frac{P_1}{P}\right)^{1-\mu} C + \left(\frac{P_1}{P}\right)(I_1 + G_1) + (1-\mathrm{w})\left(\frac{P_2}{P}\right)^{1-\mu} C + \left(\frac{P_2}{P}\right)(I_2 + G_2)}$$

from which we arrive at *parametrization 2*:

$$\mathrm{w} = \frac{(\omega_Y - 1)\left(\frac{P_1}{P}\right)(I_1 + G_1) + \omega_Y\left(\frac{P_2}{P}\right)(I_2 + G_2) + \omega_Y\left(\frac{P_2}{P}\right)^{1-\mu}}{C\left[(1-\omega_Y)\left(\frac{P_1}{P}\right)^{1-\mu} + \omega_Y\left(\frac{P_2}{P}\right)^{1-\mu}\right]} \tag{64}$$

so now the model is re-parameterized in terms of the dimensionless quantity ω_Y.

This completes an equilibrium defined in terms of *dimensionless* parameters ϱ, σ_c, ψ_i and π_i, ω_C or ω_Y, ω_g and δ which depends on the unit of time. The model equilibrium is now completely defined in terms dimensionless parameters apart from the ratio of labour-augmenting parameters $\frac{\bar{A}_{2,t}}{\bar{A}_{1,t}} = \frac{\bar{A}_{2,0}}{\bar{A}_{1,0}}$ along a bgp. At $t = 0$ using the standard normalization 1, we can choose units of labour and capital so that 1 unit of each (whatever our choice) produces 1 unit of output in both sectors. Therefore we can choose $\bar{A}_{1,0} = \bar{A}_{2,0} = 1$ and the model is now complete.

An alternative choice of units would get all prices $P_1 = P_2 = P = 1$ in the steady-state and to choose the relative efficiency $\frac{\bar{A}_{2,0}}{\bar{A}_{1,0}}$ so as to match $\frac{P_1 C_1}{PC}$ or $\frac{P_1 Y_1}{PY}$ with data. But as with the non-normalized CES production function, this throws away the ability to carry out comparative statics on the steady state that results in changes in the relative price for a given exogenous relative efficiency. This is demonstrated in our illustrative simulations. We assume that sector two is more labour intensive with a choice $\alpha_1 = 0.5$, $\alpha_2 = 0.8$.[20] Then Figure 2 uses parametrization 1 and plots the parameter w, the relative

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[19] In fact for $\mu \neq 1$, the dimensions of $\mathrm{w} \in f\left(\frac{R_{y1}}{R_{y2}}\right)$ where R_{y_i} is the dimension of output in sector i.

[20] Other parameters are as before.

price in sector 2 P_2/P, the steady state of employment share n_1 and the output share as $\mu \in [0,1]$ varies between the full range of possible values. By looking at the steady state of employment share n_1 and w, we can see how, unlike the case of the one-sector model, the steady-state equilibrium actually changes with μ. Finally Figure 3 uses parametrization 2. Now n_1 is independent of μ,[21] but again the equilibrium does change as shown by the change in the other dimensionless variable w.

3.2 The Utility Function in Deviation Form

Again as with eliminating α_n and α_k in the CES production function, we can eliminate w in the CES utility function, but only as an alternative to the first parametrization. In this case write the latter in deviation form about its steady state as

$$\begin{aligned}
\frac{C_t}{\bar{C}_t} &= \left[\frac{\mathrm{w}^{1-\phi}C_{1,t}^{\phi} + (1-\mathrm{w})^{1-\phi}C_{2,t}^{\phi}}{\mathrm{w}^{1-\phi}\bar{C}_{1,t}^{\phi} + (1-\mathrm{w})^{1-\phi}\bar{C}_{2,t}^{\phi}}\right]^{\frac{1}{\phi}} \\
&= \left[\frac{\mathrm{w}^{1-\phi}\bar{C}_{1,t}^{\phi}\left(\frac{C_{1,t}}{\bar{C}_{1,t}}\right)^{\phi}}{\mathrm{w}^{1-\phi}\bar{C}_{1,t}^{\phi} + (1-\mathrm{w})^{1-\phi}\bar{C}_{2,t}^{\phi}} + \frac{(1-\mathrm{w})^{1-\phi}\bar{C}_{2,t}^{\phi}\left(\frac{C_{2,t}}{\bar{C}_{2,t}}\right)^{\phi}}{\mathrm{w}^{1-\phi}\bar{C}_{1,t}^{\phi} + (1-\mathrm{w})^{1-\phi}\bar{C}_{2,t}^{\phi}}\right]^{\frac{1}{\phi}} \\
&= \left[\omega_C\left(\frac{C_{1,t}}{\bar{C}_{1,t}}\right)^{\phi} + (1-\omega_C)\left(\frac{C_{2,t}}{\bar{C}_{2,t}}\right)^{\phi}\right]^{\frac{1}{\phi}} \qquad (65)
\end{aligned}$$

using the first-order equations in the bgp, (58) and (59) and $\mu = \frac{1}{1-\phi}$. At first sight this seems to be a very convenient way of setting up the dynamic model that eliminates w, providing the relative consumption share $\omega_C \equiv \frac{P_1 C_1}{PC} = \frac{P_1 \bar{C}_1}{P\bar{C}}$ can be calibrated from data. However we still need to quantify w because of the Dixit-Stiglitz aggregate price given by (45). Moreover if there is only data on the relative output share $\omega_Y \equiv \frac{P_1 Y_1}{PY} = \frac{P_1 \bar{Y}_1}{P\bar{Y}}$,[22] then even (65) is of little use. We conclude that the model must be set up using either re-parametrization 1 or 2 depending on the data available for the calibration.

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[21] This can be shown from the bgp steady state. From (58) we have that $\frac{Y_1}{Y_2} = \frac{N_1 (K_1/Y_1)^{\frac{1-\alpha_1}{\alpha_1}}}{N_2 (K_2/Y_2)^{\frac{1-\alpha_2}{\alpha_2}}}$. From (58) and (58) it can be shown that $h_1 = h_2$ and hence $\frac{N_1}{N_2} = \frac{n_1}{(1-n_1)}$. With Cobb-Douglas technology $\frac{K_i}{Y_i} = \frac{(\alpha_{k,i})}{R+\delta}$ is independent of μ. It follows that n_1 is also independent of μ.

[22] This is the case in Gabriel *et al.* (2010) which is a two-sector model involving a formal and an informal sector.

3.3 Linearization

The linearization confirms that the model properties depend on w which cannot be bypassed as we did for the parameter α in the one-sector model. As before define lower case variables $x_t = \log \frac{X_t}{\bar{X}_t}$ if X_t has a long-run trend or $x_t = \log \frac{X_t}{X}$ otherwise where X is the steady-state value of a non-trended variable. For variables $n_{i,t}, i = 1, 2$ define $\hat{x}_t = \log \frac{x_t}{x}$. Define the terms of trade for the two sectors by $\tau_t = \log \frac{P_{2,t}}{P_2} - \log \frac{P_{1,t}}{P_1} \equiv p_{2,1} - p_{1,t}$. Our linearized model about the BGP zero-inflation steady state then takes the state-space form

$$\begin{aligned}
k_{i,t} &= \frac{1-\delta_1}{1+g} k_{i,t-1} + \frac{\delta+g}{1+g} i_{i,t}\,; \quad i = 1, 2 \\
E_t[\lambda_{C,t+1}] &= \lambda_{C,t} - E_t[r_{t+1}]
\end{aligned}$$

$$\begin{aligned}
\lambda_{C,t} &= -(1 + (\sigma_c - 1)(1-\varrho))c_t + \varrho(\sigma_c - 1)(n_1 l_{1,t} + (1-n_1) l_{2,t}) \\
\lambda_{h_i,t} &= -(\sigma_c - 1)(1-\varrho)c_t + (1 + \varrho(\sigma_c - 1)) \frac{h_i}{1-h_i} h_{i,t}\,; \quad i = 1, 2 \\
w_t - p_t &= \lambda_{h_1,t} - \lambda_{C,t} = \lambda_{h_2,t} - \lambda_{C,t}
\end{aligned}$$

$$\begin{aligned}
c_{1,t} &= c_t + \mu(1 - \mathrm{w})\tau_t \\
c_{2,t} &= c_t - \mu \mathrm{w} \tau_t \\
y_{i,t} &= \pi_i(a_{i,t} + \hat{n}_{i,t} + h_{i,t}) + (1-\pi_i)k_{i,t}\,; \quad i = 1, 2 \\
\hat{n}_{2,t} &= -\frac{n_1}{n_2}\hat{n}_{1,t} \\
y_{i,t} &= \frac{C_i}{Y_i} c_{1,t} + \frac{I_i}{Y_i} i_{1,t} + \frac{G_i}{Y_i} g_{i,t}\,; \quad i = 1, 2 \\
\frac{1+R}{R+\delta_i} r_t &= (y_{i,t} - k_{i,t})\,; \quad i = 1, 2 \\
w_t &= y_{1,t} - n_{1,t} = y_{2,t} - n_{2,t}
\end{aligned}$$

Note that equations for c_1 and c_2 imply $c_t = \mathrm{w} c_{1,t} + (1 - \mathrm{w}) c_{2,t}$. We now see that unless $\mu = 0$ and the two goods are perfect substitutes, the linearization is *not* independent of the parameter w even after re-parametrization.

3.4 Summary

As for the CES production function in the one-sector model, the utility function in the two-sector model is defined in terms of another 'share parameter' w which is not dimensionless and cannot therefore be quantified without specifying the precise units of factors and output. Now the only way to handle this problem is to re-parameterize the model by expressing w in terms of the other endogenous variables and a newly introduced parameter. For the latter we choose either the long-run consumption share (ω_C) or the output share (ω_Y) for the two sectors. Then including either of these in the vector of parameters, θ, we can express w as w=w(θ). The log-linearized form of the model still involves w. Either re-parametrization requires some further normalization (choice of units) for the steady states, but the model dynamics (given either ω_C or ω_Y) will not depend on this choice.

4 Re-parametrization in Utility Functions: Open Economy

This section sets up an open economy version of the closed economy RBC in section 1. Again, as in the previous section, we confine ourselves to the case of only labour-augmenting change and Cobb-Douglas production function. We first set up a dynamic 2-bloc model of interconnected economies. As one becomes infinitesimally small, we arrive at the small open economy. As before we identify and set dimensional constants by using a combination of re-parametrization and choice of units.

4.1 Dynamic Model

First define composite Dixit-Stiglitz (D-S) consumption and investment indices consisting of home-produced (H) and foreign (F) differentiated goods in terms of elasticities μ_C and μ_I:

$$C_t = \left[\mathrm{w}_C^{\frac{1}{\mu_C}} C_{H,t}^{\frac{\mu_C-1}{\mu_C}} + (1-\mathrm{w}_C)^{\frac{1}{\mu_C}} C_{F,t}^{\frac{\mu_C-1}{\mu_C}} \right]^{\frac{\mu_C}{\mu_C-1}} \tag{66}$$

$$I_t = \left[\mathrm{w}_I^{\frac{1}{\mu_I}} I_{H,t}^{\frac{\mu_I-1}{\mu_I}} + (1-\mathrm{w}_I)^{\frac{1}{\mu_I}} I_{F,t}^{\frac{\mu_I-1}{\mu_I}} \right]^{\frac{\mu_I}{\mu_I-1}} \tag{67}$$

The corresponding D-S price indices are

$$P_{C,t} = \left[\mathrm{w}_C (P_{H,t})^{1-\mu_C} + (1-\mathrm{w}_C)(P_{F,t})^{1-\mu_C} \right]^{\frac{1}{1-\mu_C}} \tag{68}$$

$$P_{I,t} = \left[\mathrm{w}_I (P_{H,t})^{1-\mu_I} + (1-\mathrm{w}_I)(P_{F,t})^{1-\mu_I} \right]^{\frac{1}{1-\mu_I}} \tag{69}$$

Let the proportions of these two differentiated goods produced in the home and foreign blocs be ν and $1-\nu$ respectively. Then ν and $1-\nu$ are then measures of relative size.

Weights in the consumption baskets in the two blocs are then defined by

$$\mathrm{w}_C \;=\; 1-(1-\nu)(1-\omega_C)\,; \quad \mathrm{w}_C^* = 1-\nu(1-\omega_C^*) \tag{70}$$

In (70), $\omega_C, \omega_C^* \in [0,1]$ are a parameters that captures the degree of 'bias' in the two blocs. If $\omega_C = \omega_C^* = 1$ we have autarky, while $\omega_C = \omega_C^* = 0$ gives us the case of perfect integration. In the limit, as the home country becomes small $\nu \to 0$. Hence $\mathrm{w}_C \to \omega_C$ and $\mathrm{w}_C^* \to 1$. Thus the foreign bloc becomes closed, but as long as there is some departure from perfect integration ($\omega_C > 0$), the home country continues to consume foreign-produced consumption goods. Exactly the same applies to the investment baskets where we define ω_I and ω_I^* by

$$\mathrm{w}_I \;=\; 1-(1-\nu)(1-\omega_I)\,; \quad \mathrm{w}_I^* = 1-\nu(1-\omega_I^*) \tag{71}$$

For the small open economy as $\nu \to 0$ and $\mathrm{w}_C^* \to 1$, from (70) we have that $\frac{1-\nu}{\nu}(1-\mathrm{w}_C^*) \to 1-\omega_C^*$. Similarly, $\frac{1-\nu}{\nu}(1-\mathrm{w}_I^*) \to 1-\omega_I^*$. These are scaling factors for the exports of consumption and investment goods respectively set out below.

Standard intra-temporal optimizing decisions for home consumers and firms lead to

$$C_{H,t} \;=\; \mathrm{w}_C \left(\frac{P_{H,t}}{P_{C,t}}\right)^{-\mu_C} C_t \tag{72}$$

$$C_{F,t} \;=\; (1-\mathrm{w}_C) \left(\frac{P_{F,t}}{P_{C,t}}\right)^{-\mu_C} C_t \tag{73}$$

$$I_{H,t} \;=\; \mathrm{w}_I \left(\frac{P_{H,t}}{P_{I,t}}\right)^{-\mu_I} I_t \tag{74}$$

$$I_{F,t} \;=\; (1-\mathrm{w}_I) \left(\frac{P_{F,t}}{P_{I,t}}\right)^{-\mu_I} I_t \tag{75}$$

In the small open economy we take foreign aggregate consumption and investment, denoted by C_t^* and I_t^* respectively, as exogenous processes. Define one real exchange rate as the relative aggregate consumption price $RER_{C,t} \equiv \frac{P_{C,t}^* S_t}{P_{C,t}}$ where S_t is the nominal exchange rate. Similarly define $RER_{I,t} \equiv \frac{P_{I,t}^* S_t}{P_{I,t}}$ for investment. Then foreign counterparts of the above defining demand for the export of the home goods are

$$C_{H,t}^* \;=\; (1-\mathrm{w}_C^*) \left(\frac{P_{H,t}^*}{P_{C,t}^*}\right)^{-\mu_C^*} C_t^* = (1-\mathrm{w}_C^*) \left(\frac{P_{H,t}}{P_{C,t} RER_{C,t}}\right)^{-\mu_C^*} C_t^* \tag{76}$$

$$I_{H,t}^* \;=\; \mathrm{w}_I^* \left(\frac{P_{H,t}^*}{P_{I,t}^*}\right)^{-\mu_I^*} I_t^* = \mathrm{w}_I^* \left(\frac{P_{H,t}}{P_{I,t} RER_{I,t}}\right)^{-\mu_I^*} I_t^* \tag{77}$$

where $P^*_{H,t}$, $P^*_{C,t}$ and $P^*_{I,t}$ denote the price of home consumption, aggregate consumption and aggregate investment goods in foreign currency and we have used the law of one namely $S_t P^*_{H,t} = P_{H,t}$. Again we define

$$P^*_{C,t} = \left[\mathrm{w}^*_C (P^*_{F,t})^{1-\mu^*_C} + (1-\mathrm{w}^*_C)(P^*_{H,t})^{1-\mu^*_C} \right]^{\frac{1}{1-\mu^*_C}} \tag{78}$$

and P^*_I similarly.

There are two non-contingent one-period bonds denominated in the currencies of each bloc with payments in period t, $B_{H,t}$ and $B^*_{F,t}$ respectively in (per capita) aggregate. The real prices of these bonds are given by

$$P_{B,t} \;=\; \frac{1}{1+R_t}; \quad P^*_{B,t} = \frac{1}{(1+R^*_t)} \tag{79}$$

where $B^*_{F,t}$ is the aggregate foreign asset position of the economy denominated in home currency and $P_{H,t}Y_t$ is nominal GDP.

The representative household must obey a budget constraint in real terms:

$$C_t + P_{B,t}B_{H,t} + P^*_{B,t}RER_{C,t}B^*_{F,t} + TL_t \;=\; \frac{W_t}{P_{C,t}}N_t + B_{H,t-1} + RER_{C,t}B^*_{F,t-1} + \Gamma_t \tag{80}$$

where $P_{C,t}$ is a Dixit-Stiglitz price index defined in (68), W_t is the wage rate, TL_t are lump-sum taxes net of transfers and Γ_t are dividends from ownership of firms. The intertemporal and labour supply decisions of the household are then

$$\begin{aligned} P_{B,t} &= \beta E_t \left[\frac{\Lambda_{C,t+1}}{\Lambda_{C,t}} \right] && (81) \\ P^*_{B,t} &= \beta E_t \left[\frac{\Lambda_{C,t+1}RER_{C,t+1}}{\Lambda_{C,t}RER_{C,t}} \right] && (82) \\ \frac{W_t}{P_{C,t}} &= -\frac{\Lambda_{N,t}}{\Lambda_{C,t}} && (83) \end{aligned}$$

where

$$\begin{aligned} \Lambda_{C,t} &= (1-\varrho)C_t^{(1-\varrho)(1-\sigma)-1}(1-h_t)^{\varrho(1-\sigma)} && (84) \\ \lambda_{N,t} &= -C_t^{(1-\varrho)(1-\sigma)}\varrho(1-N_t)^{\varrho(1-\sigma)-1} && (85) \end{aligned}$$

Firms use a CD production function with the same first-order conditions as in the

RBC model. Equilibrium and foreign asset accumulation is given by

$$
\begin{aligned}
Y_t &= C_{H,t} + I_{H,t} + \frac{1-\nu}{\nu}\left[C^*_{H,t} + I^*_{H,t}\right] + G_t \\
&\equiv C_{H,t} + I_{H,t} + EX^*_t + G_t && (86) \\
EX_t &= (1-\omega^*_C)\left(\frac{P_{H,t}}{P_{C,t}RER_{C,t}}\right)^{-\mu^*_C} C^*_t + (1-\omega^*_I)\left(\frac{P_{H,t}}{P_{I,t}RER_{I,t}}\right)^{-\mu^*_I} I^*_t && (87) \\
RER_{C,t} &= \frac{\left[\mathrm{w}^*_C + (1-\mathrm{w}^*_C)\mathcal{T}_t^{\mu^*_C-1}\right]^{\frac{1}{1-\mu^*_C}}}{\left[1-\mathrm{w}_C + \mathrm{w}_C\mathcal{T}_t^{\mu_C-1}\right]^{\frac{1}{1-\mu_C}}} = \frac{1}{\left[1-\omega_C + \omega_C\mathcal{T}_t^{\mu_C-1}\right]^{\frac{1}{1-\mu_C}}} && (88) \\
RER_{I,t} &= \frac{1}{\left[1-\omega_I + \omega_I\mathcal{T}_t^{\mu_I-1}\right]^{\frac{1}{1-\mu_I}}} && (89)
\end{aligned}
$$

where the terms of trade $\mathcal{T}_t \equiv \frac{P_{F,t}}{P_{H,t}}$ and we have used $\mathrm{w}^*_C = \mathrm{w}^*_I = 1$, $\mathrm{w}_C = \omega_C$ and $\mathrm{w}_I = \omega_I$ for the small open economy.

The risk-sharing condition and the foreign Euler equations are

$$
\begin{aligned}
RER_{C,t} &= \frac{\Lambda^*_{C,t}}{\Lambda_{C,t}} && (90) \\
\frac{1}{1+R^*_t} &= \beta E_t\left[\frac{\Lambda^*_{C,t+1}}{\Lambda^*_{C,t}}\right] && (91)
\end{aligned}
$$

Current account dynamics are given by

$$
\begin{aligned}
\frac{1}{(1+R^*_t)}RER_{C,t}B^*_{F,t} &= RER_{C,t}B^*_{F,t-1} + TB_t && (92) \\
TB_t &= \frac{P_{H,t}}{P_{C,t}}Y_t - C_t - \frac{P_{I,t}}{P_{C,t}}I_t - \frac{P_{H,t}}{P_{C,t}}G_t && (93)
\end{aligned}
$$

There are now two ways to close the model. First, as is standard for models of the small open economy (SOE), we can assume processes for foreign variables R^*_t, C^*_t, I^*_t and Λ^*_t are exogenous *and independent*. Along with exogenous processes for domestic shocks A_t and G_t this completes the model. The second arguably more satisfactory approach is to acknowledge that the foreign variables are interdependent and part of a model driven by the same form of shocks and policy rules as for the SOE. But here we retain the simpler first form.

4.2 Steady State

First assume *zero growth* in the steady state: $g = g^* = 0$ and non-negative inflation. We also focus exclusively on CES consumption and investment indices and assume CD technology with labour-augmenting technical change. Then we have

$$\begin{aligned}
\frac{W}{P_C} &= -\frac{\Lambda_N}{\Lambda_C} && (94)\\
\Lambda_C &= (1-\varrho)C^{(1-\varrho)(1-\sigma)-1}(1-N)^{\varrho(1-\sigma)} && (95)\\
\Lambda_L &= -C^{(1-\varrho)(1-\sigma)}\varrho(1-N)^{\varrho(1-\sigma)-1} && (96)
\end{aligned}$$

$$\begin{aligned}
1 &= \left[\mathrm{w}_C\left(\frac{P_H}{P_C}\right)^{1-\mu_C}+(1-\mathrm{w}_C)\left(\frac{P_F}{P_C}\right)^{1-\mu_C}\right]^{\frac{1}{1-\mu_C}} && (97)\\
\frac{P_H}{P_C} &= \frac{1}{[\mathrm{w}_C+(1-\mathrm{w}_C)\mathcal{T}^{1-\mu_C}]^{\frac{1}{1-\mu_C}}} && (98)\\
C_H &= \mathrm{w}_C\left(\frac{P_H}{P_C}\right)^{-\mu_C}C && (99)\\
C_F &= (1-\mathrm{w}_C)\left(\frac{P_F}{P_C}\right)^{-\mu_C}C && (100)\\
{C_H}^* &= (1-\mathrm{w}_C^*)\left(\frac{P_H}{P_C RER_C}\right)^{-\mu_C^*}C^* && (101)\\
Y &= K^\alpha(AL)^{1-\alpha} && (102)\\
K &= \frac{(1-\alpha)P_H Y}{(R+\delta)P_I} && (103)\\
I &= (g+\delta)K && (104)\\
I_H &= \mathrm{w}_I\left(\frac{P_H/P_C}{P_I/P_C}\right)^{-\mu_I}I && (105)\\
I_F &= (1-\mathrm{w}_I)\left(\frac{P_F/P_C}{P_I/P_C}\right)^{-\mu_I}I && (106)\\
I_H^* &= (1-\mathrm{w}_I^*)\left(\frac{P_H}{PRER}\right)^{-\mu_I^*}I^* && (107)\\
\frac{P_I}{P_C} &= \left[\mathrm{w}_I\left(\frac{P_H}{P_C}\right)^{1-\mu_I}+(1-\mathrm{w}_I)\left(\frac{P_F}{P_C}\right)^{1-\mu_I}\right]^{\frac{1}{1-\mu_I}} && (108)\\
Y &= C_H+I_H+EX_C+EX_I+G_t && (109)\\
EX_C &= C^*_{H,t}=(1-\omega^*_{C,t})\left(\frac{P_H}{P_C RER_C}\right)^{-\mu_C^*}C^* && (110)\\
EX_I &= I^*_{H,t}=(1-\omega^*_{I,t})\left(\frac{P_H}{P_I RER_I}\right)^{-\mu_I^*}I^* && (111)\\
RER_C &= \frac{1}{[1-\mathrm{w}_C+\mathrm{w}_C\mathcal{T}^{\mu_C-1}]^{\frac{1}{1-\mu_C}}} && (112)\\
1 &= \beta(1+R^*) && (113)
\end{aligned}$$

The problem now is that there are n variables but only n-1 state equations! The model is only complete if we pin down the steady state of the foreign assets or equivalently the trade balance. *In other words there is a unique model associated with any choice of the long-run assets of our SOE.*

Our missing equation is therefore the *trade balance* in the steady state

$$P_C TB = P_H Y - P_C C - P_I I - P_H G = \underbrace{P_H EX_C - (P_C C - P_H C_H)}_{\text{Net Exports of C-goods}} + \underbrace{P_H EX_I - (P_I I - P_H I_H)}_{\text{Net Exports of I-goods}} \tag{114}$$

using (109), for some choice of TB.

Finally we re-parameterize dimensional constants w_C, w_I, ω_C and ω_I using dimensionless trade ratios from trade data. From (114) we have

$$\begin{aligned}
{}_{imp} &\equiv \frac{\text{C-imports}}{\text{GDP}} = \frac{P_F C_F}{P_C Y} = c_y(1-\omega_C)\left(\frac{P_F}{P_C}\right)^{1-\mu_C} && (115)\\
is_{imp} &\equiv \frac{\text{I-imports}}{\text{GDP}} = \frac{P_F I_F}{P_C Y} = i_y(1-\omega_I)\frac{P_F}{P_C}\left(\frac{P_F}{P_I}\right)^{-\mu_I} && (116)\\
cs_{exp} &\equiv \frac{\text{C-exports}}{\text{GDP}} == \frac{P_H C_H^*}{P_C Y} = (1-\omega_C^*)\left(\frac{P_H}{P_C RER_C}\right)^{-\mu_C^*} c_y^* \frac{P_H Y^*}{P_C Y} && (117)\\
is_{exp} &\equiv \frac{\text{I-exports}}{\text{GDP}} = \frac{P_H I_H^*}{P_C Y} = (1-\omega_I^*)\left(\frac{P_H}{P_I RER_I}\right)^{-\mu_I^*} i_y^* \frac{P_H Y^*}{P_C Y} && (118)\\
tb &\equiv \frac{TB}{Y} = cs_{exp} + is_{exp} - cs_{imp} - is_{exp} && (119)
\end{aligned}$$

where we define dimensionless share parameters $c_y = \frac{C}{Y} = \frac{P_H C}{P_H Y}$ and c_y^*, i_y and i_y^* similarly. (115) – (118) can now be used to re-parameterize the dimensional constants ω_C, ω_I, ω_C and ω_I. But given tb, only three out of the four share ratios cs_{exp}, is_{exp}, cs_{imp} and is_{imp} are independent. We therefore need to introduce a further dimensionless observed parameter. We choose this to be the per capita GDP ratio

$$k \equiv \frac{P_F Y^*}{P_H Y} \tag{120}$$

The remaining dimensional constants are labour-augmenting change A and exogenous steady-state values of Y^*, C^* and I^*. We can put $C^* = c_y Y^*$ and $I^* = i_y Y^*$, so the only dimensional constants left are A and Y^*. We put $A = Y^* = 1$ as before by a suitable choice of units which do not need to be made explicit. This completes the choice of dimensional constant parameters in the model by a combination of convenient choice of units and the introduction of new and readily observed dimensionless parameters consisting of trade, consumption and investment shares.

Would it be more convenient to set all steady-state prices to be unity - i.e., $P_F = P_H = P_I = REC_C = RER_I = 1$ which then makes w_C, w_I, ω_C and ω_I dimensionless? This requires a choice of $\frac{A}{Y^*}$ and imposes another choice of units so that one unit of exports and imports is exchanged for one unit of home currency. As before with non-normalized CES functions, the problem with this is that if we wish to carry out comparative statics on the steady state or examine a permanent shock that shifts the economy to a new steady state, the terms of trade shifts have disappeared. Also if we were to utilize data on the terms of trade to estimate the model, say by Bayesian methods, the choice of a unitary price normalization would inevitably be inconsistent with this data.[23] Figure 4 illustrates this point by showing how the terms of trade and/or the relative income k change with the steady-state trade balance for a given $\frac{A}{Y^*}$ which reflects the relative efficiency of the SOE compared with the rest of the world. To accommodate a higher trade balance in the long run or lower income relative to the rest of the world (a higher k) the terms of trade (the relative import price) must rise.

Finally we briefly generalize our analysis to a *non-zero* balanced steady-state growth path. The bgp of the model economy with or without investment costs is now given by

$$\frac{\bar{\Lambda}_{C,t+1}}{\bar{\Lambda}_{C,t}} \equiv 1 + g_{\Lambda_C} = \left[\frac{\bar{C}_{t+1}}{\bar{C}_t}\right]^{(1-\varrho)(1-\sigma)-1} = (1+g)^{((1-\varrho)(1-\sigma)-1)} \tag{121}$$

Thus from (81)

$$1 + R = \frac{(1+g)^{1+(\sigma-1)(1-\varrho)}}{\beta} \tag{122}$$

Similarly for the foreign bloc

$$1 + R^* = \frac{(1+g^*)^{1+(\sigma^*-1)(1-\varrho^*)}}{\beta^*} \tag{123}$$

It is then possible to have different preferences and growth rates provided

$$\frac{1+R}{1+R^*} \quad = \quad \phi\left(\frac{RER_C B}{P}\right) = \frac{\Pi\beta^*}{\Pi^*\beta}\frac{(1+g)^{1+(\sigma-1)(1-\varrho)}}{(1+g^*)^{1+(\sigma^*-1)(1-\varrho^*)}} \tag{124}$$

where $\phi\left(\frac{RER_C B}{P}\right)$, $\phi' < 0$, is a risk premium. This pins down the assets in the steady state.

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[23] For instance if following León-Ledesma *et al.* (2010) we were to use the sample mean of the terms of trade to estimate the steady state then this would not result in a unitary outcome.

4.3 Summary

We now summarize the details of the treatment of dimensional constants that unifies our solution for all three models. In the RBC model we now assume only labour-augmenting change to enable a comparison with the other two models that assume CD technology. Below we define $\bar{A}_0 \equiv \overline{ZN}_0$. In all cases there is one dimensional parameter δ, the depreciation rate, that only depends on time, a unit that is specified. Distribution parameters α_n, α_k in the CES production function in the one-sector RBC model, w in the utility function of the two-sector RBC model and w_C, ω_C^*, ω_I, ω_I^* for the small open economy can be expressed in terms of the original dimensionless parameters and δ and new dimensionless share parameters. This leaves a simple normalization of output and efficiency parameters that does not require the specification of units to complete the model set-up.

Model	RBC	Two-Sector	Open Economy
CES Function	Production	C Index	C and I Indices
Dimensional Constants	A_0, α_n, α_k, δ	$A_{1,0}, A_{2,0}$,w, δ	ω_C, ω_C^*, ω_I, ω_I^*, A, Y^*, δ
New Dimensionless Constants	Wage Share	C or Y Sector Shares	Trade Shares, $k \equiv \frac{P_F Y^*}{P_H Y}$
Choice of Units	$\overline{Y}_0 = \overline{A}_0 = 1$	$\overline{A}_{1,0} = \overline{A}_{2,0} = 1$	$\overline{A}_0 = Y^* = 1$

Table 1. Summary

5 Conclusions

This paper builds up on a quite recent, but very rapidly growing, literature about the normalization of CES function in macroeconomics. Although this type of function was already used in the middle of the previous century it has been left aside in some areas of Macroeconomics during the past decades. We start from recent works on the CES production function and macro models of the business cycle and study in depth the concept of normalization of such functions in order to avoid dimensionality problems coming from the choice of units when defining inputs and output of production. We also extend the discussion also to CES utility functions in multi-sectoral and open economy models.

Our contributions regard the clarification of the normalization issue which is usually presented in the literature as a technical procedure without any appeal to dimensional analysis. We propose an alternative and equivalent way of resolving the problem called 're-parametrization' and we show that in the case of CES utility function in a two-sector model and an open economy model 're-parametrization' is the only solution. Indeed the 're-parametrization' approach proves to be equivalent, easier to implement and more general than the usual normalization procedure. For both the non-linear and linearization set-ups we show that we cannot by-pass the need to express the dimensional 'share parameter' in the utility function in terms of the remaining parameter which are either dimensionless or have a time-interval dimension.

Finally one particular avenue for future research is suggested by our application to the open economy. León-Ledesma *et al.* (2010) have demonstrated the importance of using a normalized CES production functions for estimation, especially from a Bayesian perspective. Our analysis suggests this could also be true for a Bayesian estimation of an open-economy DSGE model with a CES utility function of domestic and imported goods. Then utilizing our re-parametrization approach, data on terms of trade and trade shares could be used without losing important effects of the former in the vicinity of the steady state. As in León-Ledesma *et al.* (2010), monte-carlo methods would then indicate the importance, or otherwise, of adopting normalized CES utility functions for empirical work on the open economy.

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A Log-Linearization of the CES production function

Dropping the labor augmenting technology shock, A_t for simplicity:

$$Y_t^{\psi} \quad = \quad \alpha_k K_t^{\psi} + \alpha_n N_t^{\psi} \tag{A.1}$$

Define lower case variables $x_t = \log \frac{X_t}{X}$ where X is the bgp stationarized steady state value of a trended variable. Then

$$y_t = \alpha_k \left(\frac{K}{Y}\right)^{\psi} k_t + \alpha_n \left(\frac{N}{Y}\right)^{\psi} n_t \tag{A.2}$$

where $Y^{\psi} = \alpha_k K^{\psi} + \alpha_n N^{\psi}$. Substituting this expression for Y^{ψ} and after some manipulation we get:

$$y_t = \left(1 + \frac{\alpha_n}{\alpha_K}\left(\frac{K}{N}\right)^{\psi}\right) k_t + \left(\frac{\alpha_k}{\alpha_n}\left(\frac{L}{K}\right)^{\psi} + 1\right) n_t \tag{A.3}$$

Using the re-parametrization result in (32) and (32) we can substitute in the previous expression:

$$\frac{\alpha_k}{\alpha_n} = \frac{\pi}{1-\pi}\left(\frac{K}{N}\right)^{\psi}$$

and we obtain

$$y_t = (1-\pi)k_t + \pi n_t \tag{A.4}$$

The log-linearization of the first order conditions of the firm's problem ((18) and (19)) follows straightforwardly.

B Figures

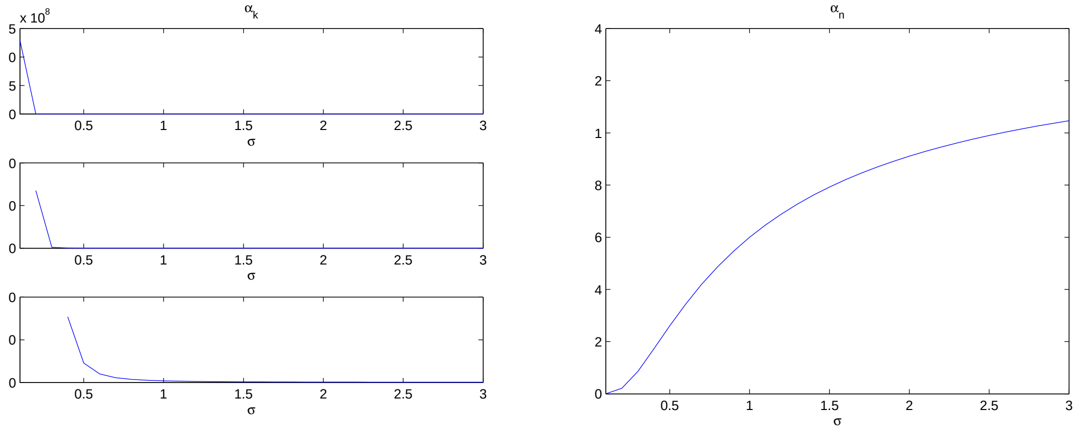

Figure 1: α_k **and** α_n **as** σ **varies**

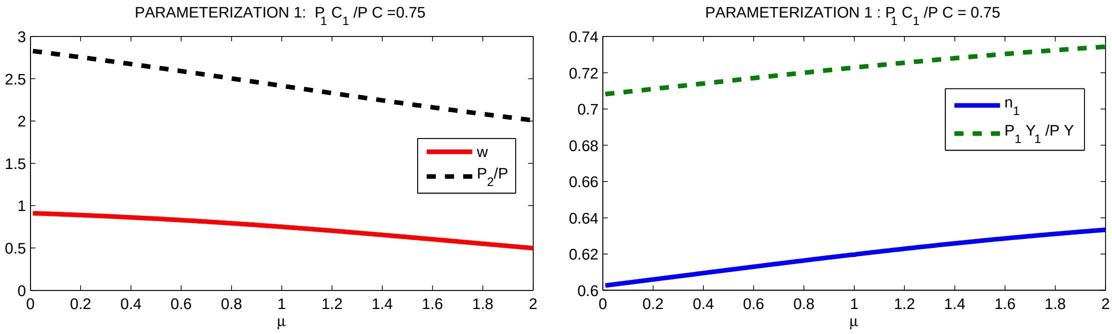

Figure 2: **Parametrization 1: Steady State Equilibrium as μ varies**

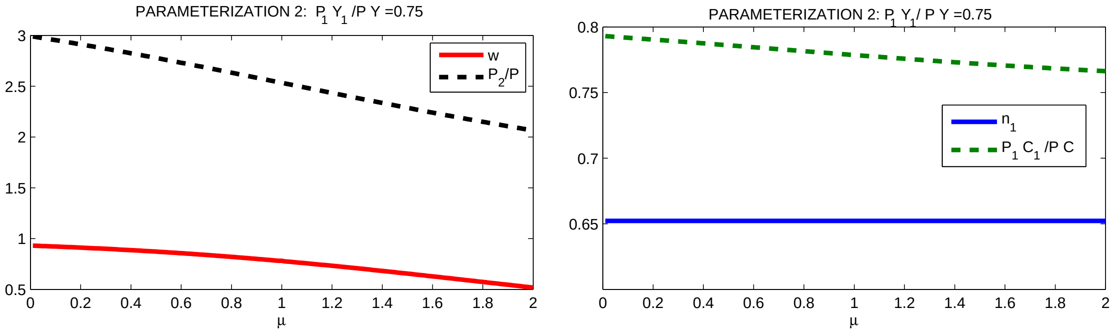

Figure 3: **Parametrization 2: Steady State Equilibrium as μ varies**

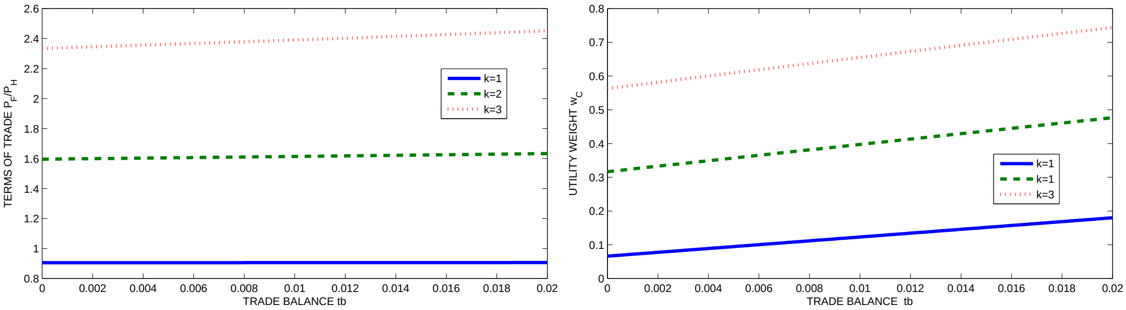

Figure 4: **Steady State Equilibrium as tb and $k \equiv \frac{P_F Y^*}{P_H Y}$ vary**