KALMAN FILTERING AND SMOOTHING IN DYNARE

Introduction

"Filtering and Smoothing of State Vector for Diffuse State Space Models", S.J. Koopman and J. Durbin (2003, in *Journal of Time Series Analysis*, vol. 24(1), pp. 85-98).

"Fast Filtering and Smoothing for Multivariate State Space Models", S.J. Koopman and J. Durbin (2000, in *Journal of Time Series Analysis*, vol. 21(3), pp. 282-296).

The State-Space Model[1]:

$$y_t = Z\alpha_t + \varepsilon_t$$

$$\alpha_{t+1} = T\alpha_t + R\eta_t$$

with:

$$\alpha_1 = a + A\delta + R_0\eta_0$$

$m \times q$ matrix A and $m \times (m-q)$ matrix R_0 are selection matrices (their columns constitue all the columns of the $m \times m$ identity matrix) so that $A'R_0 = 0$ and $A'\alpha_1 = \delta$. We assume that the vector δ is distributed as a $\mathcal{N}(0, \kappa I_q)$ for a given $\kappa > 0$. So that the expectation of α_1 is a and its variance is P, with

$$P = \kappa P_\infty + P_\star$$

$$P_\infty = AA'$$

$$P_\star = R_0 Q_0 R_0'$$

P_∞ is a $m \times m$ diagonal matrix with q ones and $m-q$ zeros. and where: y_t is a $pp \times 1$ vector, α_t is a $mm \times 1$ vector, ε_t is a $pp \times 1$ multivariate random variable (iid $\mathcal{N}(0, H)$), η_t is a $rr \times 1$ multivariate random variable (iid $\mathcal{N}(0, Q)$), a_1 is a $mm \times 1$ vector, Z_t is a $pp \times mm$ matrix, T is a $mm \times mm$ matrix, H is a $pp \times pp$ matrix, R is a $mm \times rr$ matrix, Q is a $rr \times rr$ matrix and P_1 is a $mm \times mm$ matrix.

[1]Note that in Dynare, matrices T, Z, R, H and Q are assumed to be time invariant.

1. Filtering

The filtering equations are given by:

$$\begin{aligned}
v_t &= y_t - Za_t \\
F_t &= ZP_tZ' + H \\
K_t &= P_tZ'F_t^{-1} \\
a_{t+1} &= T(a_t + K_tv_t) \\
P_{t+1} &= T(P_t - P_tZ'K')T' + RQR'
\end{aligned} \tag{1.1}$$

$\{F_t\}$ and $\{v_t\}$ are used to evaluate the likelihood. A potentially faster algorithm (unfortunately not with matlab) is to consider a univariate approach to the multivariate Kalman filter (the covariance matrix associated to the measurement errors has to be diagonal: $H = diag(\sigma_1^2, \dots, \sigma_{pp}^2)$). Let Z_i be line i of the selection matrix Z. The univariate algorithm is as follows :

$$\begin{aligned}
v_{t,i} &= y_{t,i} - Z_ia_t^{(i)} \\
F_t^{(i)} &= Z_iP_t^{(i)}Z_i' + \sigma_i^2 \\
K_t^{(i)} &= P_t^{(i)}Z_i' \\
a_t^{(i+1)} &= a_t^{(i)} + K_t^{(i)}v_{t,i}/F_t^{(i)} \\
P_t^{(i+1)} &= P_t^{(i)} - K_t^{(i)}\ {K_t^{(i)}}'/F_t^{(i)} \\
a_{t+1}^{(1)} &= Ta_t^{(pp)} \\
P_{t+1}^{(1)} &= TP_t^{(pp)}T' + RQR'
\end{aligned} \tag{1.2}$$

when $F_t^{(i)}$ is equal to zero we simply have $a_t^{(i+1)} = a_t^{(i)}$ and $P_t^{(i+1)} = P_t^{(i)}$. The log-likelihood is evaluated as follows:

$$\mathcal{L}_T = const - \frac{1}{2}\sum_{i=1}^{pp}\sum_{t=1}^{n}\log F_t^{(i)} + v_{t,i}^2/F_t^{(i)} \tag{1.3}$$

The diffuse filtering equations are given by:

$$\begin{aligned}
v_t &= y_t - Za_t \\
F_{\infty,t} &= ZP_{\infty,t}Z' + H \\
K_{\infty,t} &= P_{\infty,t}Z'F_{\infty,t}^{-1} \\
F_{*,t} &= ZP_{*,t}Z' + H \\
K_{*,t} &= (P_{*,t}Z' - K_{\infty,t}F_{*,t})\,F_{\infty,t}^{-1} \\
P_{*,t+1} &= T(P_{*,t} - P_{*,t}Z'K_{\infty,t}' - P_{\infty,t}Z'K_{*,t}')T' + RQR' \\
P_{\infty,t+1} &= T(P_{\infty,t} - P_{\infty,t}Z'K_{\infty,t}')T' \\
a_{t+1} &= T(a_t + K_{\infty,t}v_t)
\end{aligned} \tag{1.4}$$

When the condition $rank(P_{\infty,t+1}) = 0$ is satisfied we set $d = t$ and go back to the standard Kalman filtering equations. Here $F_{\infty,t}$ is assumed to be a full rank matrix. If this is not the case we switch to another algorithm. If $F_{\infty,t} = 0$:

$$
\begin{aligned}
v_t &= y_t - Za_t \\
F_{*,t} &= ZP_{*,t}Z' + H \\
K_{*,t} &= P_{*,t}Z'F_{\infty,t}^{-1} \\
P_{*,t+1} &= T(P_{*,t} - P_{*,t}Z'K'_{*,t})T' + RQR' \\
P_{\infty,t+1} &= TP_{\infty,t}T' \\
a_{t+1} &= T(a_t + K_{*,t}v_t) \\
L_t &= T(I - K_tZ)
\end{aligned}
\tag{1.5}
$$

otherwise, we consider a diffuse version of the univariate approach described above.

2. Smoothing

The smoothing equations are given by:

$$
\begin{aligned}
r_{t-1} &= Z'F_t^{-1}v_t + L'_tr_t \\
\widehat{\alpha}_t &= a_t + P_tr_{t-1} \\
\widehat{\eta}_t &= QRr_t \\
\widehat{\varepsilon}_t &= H\left(F_t^{-1}v_t - K'_tr_t\right)
\end{aligned}
\tag{2.1}
$$

initializing with $r_n = 0$ and with $L_t = T - K_tZ$. The diffuse smoothing equations are given by :

$$
\begin{aligned}
r_{t-1}^{(0)} &= L_{\infty,t}r_t^{(0)} \\
r_{t-1}^{(1)} &= Z'F_{\infty,t}^{-1}v_t - K'_{*,t}r_t^{(0)} + L'_{\infty,t}r_t^{(1)} \\
\widehat{\alpha}_t &= a_t + P_{*,t}r_{t-1}^{(0)} + P_{\infty,t}r_{t-1}^{(1)} \\
\widehat{\eta}_t &= QRr_t^{(0)} \\
\widehat{\varepsilon}_t &= -HK'_{\infty,t}r_t^{(0)}
\end{aligned}
\tag{2.2}
$$

for $t = d, d-1, ..., 1$, where d is such that $P_{\infty,d+1} = 0$. This backward recurrence is initialized with $r_d^{(0)} = r_d$, obtained from the non diffuse Kalman smoother, and $r_d^{(1)} = 0$. $L_{\infty,t} = T - K_{\infty,t}Z$.

A univariate smoothing algorithm has to be coded... In the smoothing part the matrix F_t (or $F_{\infty,t}$) is assumed to be full rank...