Solving rational expectations models at first order: what Dynare does*

Sébastien Villemot†

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Abstract

This paper describes in detail the algorithm implemented in Dynare for computing the first order approximated solution of a nonlinear rational expectations model. The core of the algorithm is a generalized Schur decomposition (also known as the QZ decomposition), as advocated by several authors in the literature. The contribution of the present paper is to focus on implementation details that make the algorithm more generic and more efficient, especially for large models.

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1 Introduction

Perturbation techniques are widely used for solving and estimating rational expectation models and Dynare[1] offers a popular, user-friendly access to these techniques. The purpose of the present paper is to describe in detail the algorithm implemented in Dynare for computing the first order approximated solution of nonlinear rational expectations models.[2]

This algorithm is based on a generalized Schur decomposition—also known as the QZ decomposition—and is therefore essentially a variation on the methods presented by Klein (2000), Sims (2002) and Uhlig (1999).

The contribution of this paper is to present some implementation details that make the algorithm more generic and more efficient for large models. In particular I describe the algorithm for removing the leads and lags of more than one in a nonlinear model. I also describe a way of reducing the size of the Schur decomposition problem by dealing separately with endogenous variables that appear only at the current date (called *static* endogenous variables in the following).

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†Paris School of Economics and CEPREMAP. E-mail: sebastien@dynare.org.

[1]Dynare is a free software platform for handling a wide class of economic models. See http://www.dynare.org and Adjemian et al. (2011) for more details.

[2]This algorithm is available using the `stoch_simul` command of Dynare. The original implementation of this algorithm was done by Michel Juillard, using MATLAB, and is available in the `matlab/dyn_first_order_solver.m` file which is distributed with Dynare.

It should be noted that Dynare is able to go further than first order and can deliver second and third order approximation of the solution of rational expectations models. These higher order solutions can be computed recursively using the first order solution as a starting point. For algorithmic details on this issue, the interested reader can refer to Collard and Juillard (2001) or Schmitt-Grohé and Uríbe (2004).

The rest of this paper is organized as follows. Section 2 presents the class of models to be solved and defines a typology of the endogenous variables. Section 3 defines the solution to the model and characterizes its first order approximation. Sections 4 and 5 describe the algorithm used to recover this first order approximation.

2 The problem to be solved

2.1 The model

In the following, we consider a dynamic nonlinear rational expectations model composed of several variables (endogenous and exogenous) and several equations. In compact form, the model is written as:

$$\mathbb{E}_t f(y_{t+1}^+, y_t, y_{t-1}^-, u_t) = 0 \tag{1}$$

where y_t is the vector of endogenous variables, y_{t+1}^+ (resp. y_{t-1}^-) is the subset of variables of y_t that appear with a lead (resp. a lag), and u_t is the vector of exogenous variables. For obvious identification reasons, the model must contain as many equations as there are endogenous variables; let n be this number.

For the timing of variables, the convention used here is the same as in Dynare: a variable decided at date t should have a time subscript of t. For example, in a typical RBC model, the capital stock used to produce date t output is actually decided at date $t-1$, so it will be written as k_{t-1} using this convention. Thus accordingly, the law of motion of capital will be $k_t = (1-\delta)k_{t-1} + i_t$. Another way of expressing this timing convention is that stock variables should use the "end-of-period" convention.

The vector of exogenous variables u_t (of dimension p) follows a Markov process:

$$u_t = P(u_{t-1}, \varepsilon_t)$$

where the ε_t are independent and identically distributed innovations with zero mean and variance-covariance matrix Σ.

Note that the stochastic process adopted here for exogenous variables is more generic than the one allowed in Dynare (which only accepts a white noise for stochastic variables, *i.e.* $u_t = \varepsilon_t$).

2.2 Typology of variables

All endogenous variables are required to appear at least at one period. However it is not required that all endogenous variables appear at the current period (a weaker condition is actually sufficient, see assumption (5) below).

We define four types of endogenous variables:

Static endogenous variables: those that appear only at the current period. Their number is $n^s \leq n$, and their indices in the y_t vector are $\zeta_j^s, j = 1 \ldots n^s$

Purely forward endogenous variables: those that appear only at the future period, possibly at the current period, but not at the previous period. Their number is $n^{++} \leq n$, and their indices $\zeta_j^{++}, j = 1 \ldots n^{++}$

Purely backward endogenous variables: those that appear only at the previous period, possibly at the current period, but not at the future period. Their number is $n^{--} \leq n$, and their indices $\zeta_j^{--}, j = 1 \ldots n^{--}$

Mixed endogenous variables: those that appear both at the future and the previous period, and possibly at the current period. Their number is $n^m \leq n$, and their indices $\zeta_j^m, j = 1 \ldots n^m$

These four types of variables form a partition of the endogenous variables, and we therefore have:

$$n^m + n^{++} + n^{--} + n^s = n$$

We also define:

Forward endogenous variables: the union of mixed and purely forward endogenous variables. Their number is $n^+ = n^{++} + n^m$, and their indices $\zeta_j^+, j = 1 \ldots n^+$.

Backward endogenous variables: the union of mixed and purely backward endogenous variables. Their number is $n^- = n^{--} + n^m$, and their indices $\zeta_j^-, j = 1 \ldots n^-$

Dynamic endogenous variables: all the variables except static endogenous variables. Their number is $n^d = n - n^s$, and their indices $\zeta_j^d, j = 1 \ldots n^d$

The seven indices are such that $1 \leq \zeta_1^k < \zeta_2^k < \ldots < \zeta_{n^k}^k \leq n$, where $k \in \{s, +, ++, -, --, m, d\}$.

We denote by $y_t = (y_{1,t}, \ldots, y_{n,t})'$ the vector of endogenous variables at date t. We denote by $y_t^k = (y_{\zeta_1^k,t}, \ldots, y_{\zeta_{n^k}^k,t})'$ a subvector of endogenous variables, where $k \in \{s, +, ++, -, --, m, d\}$.

We denote by $\beta_j^+, j = 1 \ldots n^m$ the indices of mixed endogenous variables inside the ζ_j^+ sequence, *i.e.* β_j^+ is such that $\zeta_{\beta_j^+}^+$ is a mixed endogenous variable. We similarly define β_j^- for mixed endogenous variables inside the ζ_j^- sequence. We similarly define π_j^+ (resp. π_j^-) for purely forward (resp. purely backward) endogenous variables inside ζ_j^+ (resp ζ_j^-).

Finally, the vector of *state variables* is formed by the union of backward endogenous variables at the previous period and of exogenous variables at the current period, and is therefore of size $n^- + p$.

2.3 Removing extra leads and lags

The form given in equation (1) makes the assumption that endogenous variables appear with at most one lead and one lag, and that exogenous variables appear only at the current period. This assumption does not imply any loss of generality, since it is easy to transform a nonlinear model with many leads and lags into an equivalent model of the form given in (1), as is detailed below.[3]

For every variable x_t in the original model whose maximum lag is x_{t-d-k} with $k > 0$ (and $d = 1$ if x is an endogenous variable or $d = 0$ if it is an exogenous variable), the transformation is the following:

- introduce k new endogenous variables z_t^j, for $j \in \{1, \ldots, k\}$;

[3]The algorithm described in the present section is implemented in the Dynare preprocessor, since version 4.1. Auxiliary variables are created automatically and will show up at several places in Dynare output; see Adjemian et al. (2011) for the names of these variables.

- add k new equations to the model:

$$\begin{cases} z_t^1 = x_{t-d} \\ z_t^j = z_{t-1}^{j-1} \quad \text{for } j \in \{2, \ldots, k\} \end{cases};$$

- replace all occurrences of x_{t-d-j} (with $j > 0$) in the original model by z_{t-1}^j in the transformed model.

The transformation for variables with a lead is a bit more elaborate because one has to handle the fact that there is an expectation operator in front of all equations. The algorithm is as follows:

- decompose every equation of the original model in the following form:

$$A + \sum_{i \in I} B_i \, \mathbb{E}_t C_i = 0$$

  where A and the B_i are (possibly nonlinear) expressions containing only current or lagged variables, and the C_i are (possibly nonlinear) expressions which may contain leads; this decomposition is not unique, but one should aim at making the C_i terms as simple as possible;

- for every C_i where there is a lead of 2 or more on an endogenous variable, or a lead on an exogenous variable:
  - let k be the minimal number of periods so that $C_i^{(-k)}$ has at most one lead on endogenous variables and no lead on exogenous variables (where $C_i^{(-k)}$ stands for the transformation of C_i where all variables have been lagged by k periods);
  - introduce k new endogenous variables z_t^j, for $j \in \{1, \ldots, k\}$;
  - add k new equations to the model:

$$\begin{cases} z_t^1 = \mathbb{E}_t C_i^{(-k)} \\ z_t^j = \mathbb{E}_t z_{t+1}^{j-1} \quad \text{for } j \in \{2, \ldots, k\} \end{cases};$$

  - replace all occurrences of $\mathbb{E}_t C_i$ in the original model by $\mathbb{E}_t z_{t+1}^k$ in the transformed model.

It is straightforward to see that this transformed model is under the form given in (1). And by the law of iterated expectations, it is equivalent to the original one.

3 The solution and its first order approximation

We first define the deterministic steady state of the model as the vector $(\bar{y}, \bar{u}, \bar{\varepsilon})$ satisfying:

$$\bar{\varepsilon} = 0$$

$$\bar{u} = P(\bar{u}, \bar{\varepsilon})$$

$$f(\bar{y}^+, \bar{y}, \bar{y}^-, \bar{u}) = 0$$

Finding the deterministic steady state involves the resolution of a multivariate nonlinear system.[4] Then, finding the rational expectation solution of the model means finding the policy

[4] Dynare offers efficient ways of performing this task, but this is out of the scope of the present paper.

functions (also known as decision rules), which give current endogenous variables as a function of state variables:

$$y_t = g(y_{t-1}^-, u_t)$$

Note that, by definition of the deterministic steady state, we have $\bar{y} = g(\bar{y}^-, \bar{u})$.

The function g is characterized by the following functional equation:

$$\mathbb{E}_t f\left[g^+(g^-(y_{t-1}^-, u_t), u_{t+1}), g(y_{t-1}^-, u_t), y_{t-1}^-, u_t\right] = 0 \tag{2}$$

where g^+ (resp. g^-) is the restriction of g to forward (resp. backward) endogenous variables.

In the general case, this functional equation cannot be solved exactly, and one has to resort to numerical techniques to get an approximated solution. The remainder of this paper describes the first order perturbation technique implemented in Dynare. Let:

$$f_{y^+} = \frac{\partial f}{\partial y_{t+1}^+},\ f_{y^0} = \frac{\partial f}{\partial y_t},\ f_{y^-} = \frac{\partial f}{\partial y_{t-1}^-},\ f_u = \frac{\partial f}{\partial u_t}$$

$$g_y = \frac{\partial g}{\partial y_{t-1}^-},\ g_u = \frac{\partial g}{\partial u_t}$$

$$P_u = \frac{\partial P}{\partial u_{t-1}},\ P_\varepsilon = \frac{\partial P}{\partial \varepsilon_t}$$

where the derivatives are taken at $\bar{y}$, $\bar{u}$ and $\bar{\varepsilon}$.

The first order approximation of the policy function is therefore:

$$\hat{g}(y_{t-1}^-, u_t) = \bar{y} + g_y \hat{y}_{t-1}^- + g_u \hat{u}_t$$

where $\hat{y}_{t-1}^- = y_{t-1}^- - \bar{y}^-$, $\hat{u}_t = u_t - \bar{u}$, and g_y and g_u are unknowns at this stage.

A first order approximation of (2) around $\bar{y}$ and $\bar{u}$ gives:

$$\begin{aligned} f(\bar{y}^+, \bar{y}, \bar{y}^-, \bar{u}) + f_{y^+}[g_y^+(g_y^- \hat{y}_{t-1}^- + g_u^- \hat{u}_t) + g_u^+ \mathbb{E}_t[P_u \hat{u}_t + P_\varepsilon \varepsilon_{t+1}]] \\ + f_{y^0}(g_y \hat{y}_{t-1}^- + g_u \hat{u}_t) + f_{y^-} \hat{y}_{t-1}^- + f_u \hat{u}_t = 0 \end{aligned}$$

where g_y^+, g_y^-, g_u^-, g_u^+ are the derivatives of the restrictions of g with obvious notation. Computing the expectancy term, taking into account the property of the deterministic steady state, and reorganizing the terms, we obtain:

$$(f_{y^+} g_y^+ g_y^- + f_{y^0} g_y + f_{y^-}) \hat{y}_{t-1}^- + (f_{y^+} g_y^+ g_u^- + f_{y^+} g_u^+ P_u + f_y g_u + f_u) \hat{u}_t = 0 \tag{3}$$

In the next sections, we exploit this equation in order to recover the unknown coefficients g_u and g_y.

4 Recovering g_y

Taking into account the term multiplying $\hat{y}_{t-1}^-$, equation (3) imposes:

$$f_{y^+} g_y^+ g_y^- + f_{y^0} g_y + f_{y^-} = 0$$

This amounts to:

$$f_{y^+} \hat{y}_{t+1}^+ + f_{y^0} \hat{y}_t + f_{y^-} \hat{y}_{t-1}^- = 0 \tag{4}$$

Let S be the $n \times n^s$ submatrix of f_{y^0} where only the columns for static endogenous variables are kept, *i.e.* $S_{i,j} = f_{y^0,i,\zeta_j^s}$. A QR decomposition[5] gives $S = QR$ where Q is an $n \times n$ orthogonal matrix and R an $n \times n^s$ upper triangular matrix.

For the model to be identified, we assume that:

$$rank(R) = n^s. \tag{5}$$

Thus, equation (4) can be rewritten as:

$$A^+ \hat{y}_{t+1}^+ + A^0 \hat{y}_t + A^- \hat{y}_{t-1}^- = 0 \tag{6}$$

where $A^+ = Q' f_{y^+}$, $A^0 = Q' f_{y^0}$ and $A^- = Q' f_{y^-}$. By construction, columns of static endogenous variables in A^0 are zero in their lower part: $\forall i > n^s, \forall j \leq n^s,\ A^0_{i,\zeta_j^s} = 0$.

4.1 Non-static endogenous variables

Taking only the n^d lower rows of system (6), we get:

$$\tilde{A}^+ \hat{y}_{t+1}^+ + \tilde{A}^{0+} \hat{y}_t^+ + \tilde{A}^{0-} \hat{y}_t^- + \tilde{A}^- \hat{y}_{t-1}^- = 0 \tag{7}$$

where $\tilde{A}^+$ (resp. $\tilde{A}^-$) contains the last n^d rows of A^+ (resp. A^-). Matrices $\tilde{A}^{0+}$ and $\tilde{A}^{0-}$ can be defined in two ways, depending on where we deal with mixed endogenous variables:

- $\tilde{A}^{0+}$ is a submatrix of A^0 where only the last n^d rows and the columns for forward endogenous variables are kept ($\tilde{A}^{0+}_{i,j} = A^0_{n^s+i,\zeta_j^+}$), and $\tilde{A}^{0-}$ is such that purely backward columns are taken from A^0 ($\tilde{A}^{0-}_{i,\pi_j^-} = A^0_{n^s+i,\zeta_j^{--}}$), and the rest is zero;
- $\tilde{A}^{0-}$ is a submatrix of A^0 where only the last n^d rows and the columns for backward endogenous variables are kept ($\tilde{A}^{0-}_{i,j} = A^0_{n^s+i,\zeta_j^-}$), and $\tilde{A}^{0+}$ is such that purely forward columns are taken from A^0 ($\tilde{A}^{0+}_{i,\pi_j^+} = A^0_{n^s+i,\zeta_j^{++}}$), and the rest is zero.

Note that in equation (7), static endogenous variables no longer appear.

The structural state space representation of (7) is:

$$\underbrace{\begin{pmatrix} \tilde{A}^{0-} & \tilde{A}^+ \\ I^- & 0 \end{pmatrix}}_{D} \begin{pmatrix} \hat{y}_t^- \\ \hat{y}_{t+1}^+ \end{pmatrix} = \underbrace{\begin{pmatrix} -\tilde{A}^- & -\tilde{A}^{0+} \\ 0 & I^+ \end{pmatrix}}_{E} \begin{pmatrix} \hat{y}_{t-1}^- \\ \hat{y}_t^+ \end{pmatrix}$$

where I^- is an $n^m \times n^-$ selection matrix for mixed endogenous variables, such that $I^-_{i,\beta_i^-} = 1$, and zero otherwise. Similarly, I^+ is an $n^m \times n^+$ matrix, such that $I^+_{i,\beta_i^+} = 1$, and zero otherwise. Therefore, D and E are square matrices of size $n^{++} + n^{--} + 2n^m$.

Using the fact that $\hat{y}_{t+1}^+ = g_y^+ \hat{y}_t^-$, this can be rewritten as:

$$D \begin{pmatrix} I_{n^-} \\ g_y^+ \end{pmatrix} \hat{y}_t^- = E \begin{pmatrix} I_{n^-} \\ g_y^+ \end{pmatrix} \hat{y}_{t-1}^- \tag{8}$$

where I_{n^-} is the identity matrix of size n^-.

[5] See Golub and Van Loan (2013, section 5.2).

A generalized Schur decomposition (also known as the QZ decomposition) of the pencil (D, E) is performed:[6]

$$\begin{cases} D &= QTZ \\ E &= QSZ \end{cases}$$

where T is upper triangular, S quasi upper triangular, and Q and Z are orthogonal matrices. The decomposition is done is such a way that stable generalized eigenvalues (modulus less than 1) are in the upper left corner of T and S.

Matrices T and S are block decomposed so that the upper left block of both matrices is square and contains generalized eigenvalues of modulus less than 1, and the lower right block is square and contains generalized eigenvalues of modulus strictly greater than 1.

Equation (8) can be rewritten as:

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_{n^-} \\ g_y^+ \end{pmatrix} \hat{y}_t^- = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_{n^-} \\ g_y^+ \end{pmatrix} \hat{y}_{t-1}^- \tag{9}$$

where T_{11} and S_{11} are square and contain stable generalized eigenvalues, while T_{22} and S_{22} are square and contain explosive generalized eigenvalues.

To exclude explosive trajectories, we impose:

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_{n^-} \\ g_y^+ \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix} \tag{10}$$

which implies:

$$g_y^+ = -(Z_{22})^{-1} Z_{21}$$

Note that the squareness of Z_{22} is the Blanchard and Kahn (1980) *order* condition (*i.e.* the requirement to have as many explosive eigenvalues as forward or mixed endogenous variables), and the non-singularity of Z_{22} is the Blanchard and Kahn (1980) *rank* condition.

Using equation (10) and the fact that $\hat{y}_t^- = g_y^- \hat{y}_{t-1}^-$, equation (9) implies:

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix} g_y^- = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix}$$

Then, using the fact that solving equation (10) for X gives $X = (Z'_{11})^{-1}$, the upper part of this system gives the solution for g_y^-:

$$g_y^- = X^{-1} T_{11}^{-1} S_{11} X = Z'_{11} T_{11}^{-1} S_{11} (Z'_{11})^{-1}$$

Note that mixed variables appear in both g^+ and g^-: the corresponding lines will be equal across the two matrices by construction.

4.2 Static endogenous variables

The n^s upper lines of equation (6) can be written as:

$$\breve{A}^+ \hat{y}_{t+1}^+ + \breve{A}^{0d} \hat{y}_t^d + \breve{A}^{0s} \hat{y}_t^s + \breve{A}^- \hat{y}_{t-1}^- = 0 \tag{11}$$

where $\breve{A}^+$ (resp. $\breve{A}^-$) contains the first n^s rows of A^+ (resp. A^-). Matrix $\breve{A}^{0s}$ (resp. $\breve{A}^{0d}$) contains the first n^s rows and only the static (resp. non-static) columns of A^0. Recall that $\breve{A}^{0s}$ is a square upper triangular matrix by construction, and it is invertible because of assumption (5).

[6] See Golub and Van Loan (2013, section 7.7) for theoretical and practical details on this decomposition.

Equation (11) can be rewritten as:

$$\breve{A}^{+} g_y^{+} g_y^{-} \hat{y}_{t-1}^{-} + \breve{A}^{0d} g_y^{d} \hat{y}_{t-1}^{-} + \breve{A}^{0s} \hat{y}_t^{s} + \breve{A}^{-} \hat{y}_{t-1}^{-} = 0$$

where g_y^d, the restriction of g_y to non-static endogenous variables, is obtained by combining g_y^+ and g_y^-. We therefore have:

$$g_y^s = -\left[\breve{A}^{0s}\right]^{-1} \left(\breve{A}^{+} g_y^{+} g_y^{-} + \breve{A}^{0d} g_y^{d} + \breve{A}^{-}\right)$$

5 Recovering g_u

Equation (3) restricted to $\hat{u}_t$ imposes:

$$f_{y^+} g_y^{+} g_u^{-} + f_{y^+} g_u^{+} P_u + f_y g_u + f_u = 0,$$

and be rewritten as:

$$(f_{y^+} g_y^{+} J^{-} + f_y) g_u + f_{y^+} J^{+} g_u P_u + f_u = 0$$

where J^- (resp J^+) is an $n^- \times n$ matrix (resp. $n^+ \times n$ matrix) selecting only the backward (resp. forward) endogenous variables. In the particular case solved by Dynare, where $P_u = 0$, the solution to this equation is:

$$g_u = -(f_{y^+} g_y^{+} J^{-} + f_y)^{-1} f_u$$

In the general case, this equation is a specialized Sylvester equation, which can be solved using the algorithm proposed by Kamenik (2004)[7].

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