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Taking Perturbation to the Accuracy Frontier: A Hybrid of Local and Global Solutions*

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Abstract

Local (perturbation) methods compute solutions in one point and tend to deliver far lower accuracy levels than global solution methods. In the present paper, we develop a hybrid method that solves for some policy functions locally (using a standard perturbation method) and that solves for the other policy functions globally to satisfy certain nonlinear optimality conditions (using closed-form expressions and a numerical solver). We applied the hybrid method to solve large-scale RBC models used in the comparison analysis of Kollmann et al. (2011b). We obtained more accurate solutions than those produced by any other (either local or global) solution method participating in that comparison.

JEL classification: C63; C68; C88; F41

Keywords: Dynare; Perturbation; Hybrid; Accuracy; Numerical methods; Approximation

1 Introduction

Local (perturbation) methods compute solutions in just one point—a deterministic steady state—using Taylor expansions of optimality conditions.[1] The advantage of perturbation methods is their low computational expense. The shortcoming is that the accuracy of local solutions may decrease rapidly away from the steady state; see Judd and Guu (1993), and Kollmann et al. (2011b) for accuracy results in one-agent and multi-agent economies, respectively.[2] In turn, global methods (such as projection and stochastic simulation ones) compute solutions on larger domains, and the range of their accuracy is much wider; see Judd (1992) and Kollmann et al. (2011b) for accuracy results in one-agent and multi-agent economies, respectively.[3] How-

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[1] Perturbation methods are studied in Judd and Guu (1993), Gaspar and Judd (1997), Judd (1998), Collard and Juillard (2001), Jin and Judd (2002), Schmitt-Grohé and Uríbe (2004), Aruoba et al. (2006), Swanson et al. (2006), Chen and Zadrozny (2009) and Gomme and Klein (2011) among others.

[2] In particular, the first- and second-order perturbation methods of Kollmann et al. (2011a) produce maximum approximation errors of 6.3% and 1.35% on a stochastic simulation, and they produce maximum errors of 65% and 50% on a 30% deviation from the steady state; see Kollmann et al. (2011b).

[3] In the accuracy comparison of Kollmann et al. (2011b), global quadratic solutions are up to three orders of magnitude more accurate than local quadratic solutions.

ever, the cost of global solution methods is high and increases rapidly with the dimensionality of the state space.[4]

The present paper makes three contributions to the literature. First, we develop a perturbation-based solution method that combines local and global approximation techniques. Our hybrid method is as follows: compute the standard perturbation solution, fix some perturbation policy functions (these functions constitute the first part of the hybrid solution) and compute the remaining policy functions to satisfy certain nonlinear optimality conditions (the resulting functions constitute the second part of the hybrid solution). The construction of the second part of the hybrid solution mimics global solution methods: for each point of the state space considered, we solve nonlinear equations either analytically (when closed-form solutions are available) or with a numerical solver. If the perturbation policy functions, used to construct a hybrid solution, are accurate, then the entire hybrid solution inherits their high accuracy and can be far more accurate than the original perturbation solution. The cost of the proposed hybrid method is essentially the same as that of the standard perturbation method.

Second, we show how to apply the proposed hybrid method for solving dynamic economic models. Our numerical analysis is carried out in the context of a multi-country RBC model used in Kollmann et al. (2011b) to compare the performance of six state-of-art solution methods.[5] This model represents various challenges to numerical methods, including a large number of state variables, endogenous labor-leisure choice, heterogeneity in fundamentals and absence of closed-form expressions for endogenous variables. Our implementation of the hybrid method relies on Dynare perturbation software. We compute perturbation solutions up to the third order, we fix capital policy functions for all countries, we simulate the economy path for the state variables (capital stocks and productivity levels), and we fill-in the consumption and labor allocations of all countries by solving a system of intratemporal optimality conditions (we use a fixed-point "iteration-on-allocations" solver introduced in Maliar et al. (2011)). Our hybrid method proved to be both accurate and reliable in the application considered.

Finally, we extend the accuracy frontier attained in the related literature. Namely, our simple and low-cost hybrid method produces solutions that are more accurate than those produced by any other (either local or global) solution method participating in the comparison analysis of Kollmann et al. (2011b). In particular, the maximum error produced by our hybrid method on a stochastic simulation in an eight-country version of the model does not exceed 0.001%, which is about three times smaller than the corresponding error produced by a cluster grid algorithm (CGA) of Maliar et al. (2011) (most accurate on a stochastic simulation). The running time for our hybrid method in large-scale applications is orders of magnitude lower than that of global solution methods considered in Kollmann et al. (2011b). It takes us just 1.32 seconds to compute the most accurate third-order solution for the eight-country version of the model.

The rest of the paper is organized as follows. Section 2 introduces the idea of hybrid method in the context of a simple example. Section 3 provides a formal description of the hybrid method that combines local (perturbation) and global solutions. Section 4 assesses the performance of the hybrid method using a large-scale RBC model studied in Kollmann et al. (2011b). Section 5 concludes.

[4]For example, for a model with 16 state variables and valued leisure, the running time for finding quadratic solutions ranges from 40 minutes to nearly 8 hours for the global methods studied in Kollmann et al. (2011b). None of the methods participating in the comparison analysis of Kollmann et al. (2011b) went beyond quadratic solutions when analyzing large-scale versions of the model.

[5]The six methods participating in the comparison analysis are: first- and second-order perturbation methods of Kollmann et al. (2011a), stochastic simulation and cluster-grid algorithms of Maliar et al. (2011), Smolyak's collocation method of Malin et al. (2011), and a monomial rule Galerkin method of Pichler (2011). The latter three methods are projection methods. See Juillard and Villemot (2011) for a description of the accuracy evaluation performed.

2 Hybrid solutions: a motivating example

In this section, we expose the idea of the hybrid method in the context of a simple example—the standard one-sector neoclassical growth model.

The model. A representative agent solves

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

$$\text{s. t. } c_t + k_{t+1} = (1-\delta) k_t + a_t f(k_t), \tag{2}$$

$$\ln a_{t+1} = \rho \ln a_t + \varepsilon_{t+1}, \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma^2), \tag{3}$$

where c_t, k_t and a_t are, respectively, consumption, capital and productivity level at period t; u and f are, respectively, the utility and production functions, which are assumed to be increasing and concave; $\delta \in (0,1]$ is the depreciation rate; $\beta \in (0,1)$ is the discount factor; $\rho \in (0,1)$ and $\sigma \geq 0$ are the auto-correlation coefficient and standard deviation of the productivity level, respectively; initial condition (k_0, a_0) is given. The Euler equation of problem (1)–(3) is

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \left[1 - \delta + a_{t+1} f'(k_{t+1}) \right] \right\}. \tag{4}$$

A solution to problem (1)–(3) is given by policy functions for capital $k_{t+1} = K(k_t, a_t)$ and consumption $c_t = C(k_t, a_t)$ satisfying (4) and (2).

Benchmark solution. Assume that a numerical method delivers an approximate solution in the form of two policy functions $\widehat{K} \approx K$ and $\widehat{C} \approx C$. Assume that this solution is not sufficiently accurate for some purposes.

Hybrid method. We attempt to improve on accuracy of the benchmark solution in the following way. We fix one policy function from the benchmark solution, for example, $\widehat{K}$, and we solve for the other policy function, $\widetilde{C} \equiv \widetilde{C}\left(\widehat{K}\right)$, to satisfy the nonlinear optimality conditions taking $\widehat{K}$ as given. If the resulting new policy function is different from the benchmark one, i.e., $\widetilde{C} \neq \widehat{C}$, we have a hybrid solution.

Four hybrid solutions for our model. For the model (1)–(3), we have two optimality conditions, namely, budget constraint (2) and Euler equation (4). By considering all possible combinations of the two policy functions and the two optimality conditions, we construct four hybrid solutions $\left\{\widehat{K}, \widetilde{C}_{BC}\right\}$, $\left\{\widehat{K}, \widetilde{C}_{EE}\right\}$, $\left\{\widehat{C}, \widetilde{K}_{BC}\right\}$ and $\left\{\widehat{C}, \widetilde{K}_{EE}\right\}$ as follows:

HYB1: Fix $\widehat{K}$ and define $\widetilde{C}_{BC}$ using (2),

$$\widetilde{C}_{BC}(k_t, a_t) \equiv (1-\delta) k_t + a_t f(k_t) - \widehat{K}(k_t, a_t).$$

HYB2: Fix $\widehat{K}$ and define $\widetilde{C}_{EE}$ using (4),

$$u'\left(\widetilde{C}_{EE}(k_t, a_t)\right) = \beta E_t \left\{ u'\left[\widetilde{C}_{EE}\left(\widehat{K}(k_t, a_t), a_{t+1}\right)\right] \left[1 - \delta + a_{t+1} f'\left(\widehat{K}(k_t, a_t)\right)\right] \right\}.$$

HYB3: Fix $\widehat{C}$ and define $\widetilde{K}_{BC}$ using (2),

$$\widetilde{K}_{BC}(k_t, a_t) \equiv (1-\delta) k_t + a_t f(k_t) - \widehat{C}(k_t, a_t).$$

HYB4: Fix $\widehat{C}$ and define $\widetilde{K}_{EE}$ using (4),

$$u'\left(\widehat{C}(k_t, a_t)\right) = \beta E_t \left\{ u'\left(\widehat{C}\left(\widetilde{K}_{EE}(k_t, a_t), a_{t+1}\right)\right) \left[1 - \delta + a_{t+1} f'\left(\widetilde{K}_{EE}(k_t, a_t)\right)\right] \right\}.$$

Discussion. On the basis of our example, we make the following observations:

1. Multiple hybrid solutions can be constructed for a given benchmark solution; in our example, there are 4 hybrid solutions.

2. The hybrid method mimics global solution methods in the sense that functions $\widetilde{C}_{BC}$, $\widetilde{C}_{EE}$, $\widetilde{K}_{BC}$ and $\widetilde{K}_{EE}$ are defined to satisfy the corresponding nonlinear optimality condition globally, for any point (k_t, a_t) of the state space considered.

3. A hybrid solution can be either more or less accurate than the benchmark solution. Assume that in the benchmark solution, $\widehat{K}$ is accurate and $\widehat{C}$ is not. Then, the hybrid solutions based on $\widehat{K}$ (i.e., HYB1 and HYB2) will be more accurate, while the hybrid solutions based on $\widehat{C}$ (i.e., HYB3 and HYB4) will be less accurate than the benchmark solution.

4. Hybrid solutions can differ in cost considerably. In our example, HYB1 and HYB3 are obtained using simple closed-form expressions, while HYB2 and HYB4 are defined implicitly and are far more costly to compute.

5. Hybrid solutions can be trivial for some benchmark solutions. For example, if the policy functions $\widehat{K}$ and $\widehat{C}$ were computed by imposing budget constraint (2) exactly (as do global solution methods), then HYB1 and HYB3 will be identical to the benchmark solution.

We will not perform the numerical analysis of equilibrium in our simple model since this model can be solved both accurately and quickly by many existing solution methods. In Section 4, we will solve a challenging large-scale version of the present model.

3 Hybrid of perturbation and global solutions

The hybrid technique can potentially be used to increase accuracy of any solution method. That is, whenever some policy functions are approximated less accurately than others, we can keep the accurate policy functions and recompute the inaccurate ones. However, it is of particular interest to construct hybrid solutions for the perturbation class of methods. Perturbation methods are very cheap but their accuracy is limited. Therefore, the goal is to increase the accuracy of perturbation methods without increasing their cost.[6] In the remainder of the paper, we confine our attention to hybrids of perturbation and global solutions. Below, we outline the studied class of problems, elaborate a hybrid method that combines local and global solutions, discuss the properties of the hybrid solutions and describe the relation of our hybrid method to the literature.

3.1 The studied class of problems

We study a class of problems, whose solutions are characterized by the following set of equations for $t = 0, 1, \ldots$

$$E_t \left[H \left(\boldsymbol{x}_t, \boldsymbol{z}_t, \boldsymbol{y}_t, \boldsymbol{x}_{t+1}, \boldsymbol{z}_{t+1}, \boldsymbol{y}_{t+1} \right) \right] = \mathbf{0}, \tag{5}$$

$$G \left(\boldsymbol{x}_t, \boldsymbol{z}_t, \boldsymbol{y}_t, \boldsymbol{x}_{t+1} \right) = \mathbf{0}, \tag{6}$$

$$\boldsymbol{z}_{t+1} = \Phi \boldsymbol{z}_t + \boldsymbol{\varepsilon}_{t+1}, \tag{7}$$

[6] Global solution methods can also benefit from the hybrid technique; see Maliar et al. (2011) for examples. However, the cost of hybrid solutions constructed using global solutions will be at least as high as that of the global solutions used for their construction.

where initial condition $(\boldsymbol{x}_0, \boldsymbol{z}_0)$ is given; E_t denotes the operator of conditional expectation; $\boldsymbol{x}_t \in \mathbb{R}^{n_x}$ is a vector of endogenous state variables at t (*e.g.*, capital); $\boldsymbol{z}_t \in \mathbb{R}^{n_z}$ is a vector of exogenous state (random) variables at t (*e.g.*, productivity); $\boldsymbol{y}_t \in \mathbb{R}^{n_y}$ is a vector that contains t-period control variables (*e.g.*, consumption, labor) and other variables (*e.g.*, prices, Lagrange multipliers) given by functions of variables known at t; with a slight abuse of terminology, we refer to $\boldsymbol{y}_t$ as control variables; $\mathbf{0}$ denotes a column-vector of zeros; H and G are vector functions that are assumed to be q-times continuously differentiable; $\boldsymbol{\varepsilon}_{t+1} \in \mathbb{R}^{n_z}$ is a vector of disturbances, $\boldsymbol{\varepsilon}_{t+1} \sim \mathcal{N}(\mathbf{0}, \Sigma)$ with $\Sigma \in \mathbb{R}^{n_z \times n_z}$ being a variance-covariance matrix; $\Phi \in \mathbb{R}^{n_z \times n_z}$ determines the auto-correlation of $\boldsymbol{z}_t$ and has eigenvalues whose absolute values are less than one.[7]

Equations of type (5) include variables not known at t, $\boldsymbol{z}_{t+1}$ and $\boldsymbol{y}_{t+1}$, and require the evaluation of conditional expectations. We refer to these n_H equations as *inter-temporal-choice conditions*. Equations of type (6) contain only variables known at t. We refer to these n_G equations as *intra-temporal-choice conditions*.[8] Finally, equations of type (7) are laws of motion for exogenous state variables; there are n_z such equations.

A deterministic steady state is defined as a set of values for the endogenous variables, $\boldsymbol{x}_t = \boldsymbol{x}_{t+1} = \overline{\boldsymbol{x}}$ and $\boldsymbol{y}_t = \boldsymbol{y}_{t+1} = \overline{\boldsymbol{y}}$, that solves equations (5) and (6) in the absence of exogenous shocks (*i.e.*, $\boldsymbol{z}_t = \boldsymbol{z}_{t+1} = \mathbf{0}$, $H(\overline{\boldsymbol{x}}, \mathbf{0}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}}, \mathbf{0}, \overline{\boldsymbol{y}}) = \mathbf{0}$ and $G(\overline{\boldsymbol{x}}, \mathbf{0}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}}) = \mathbf{0}$). Let $\boldsymbol{e}_t \equiv \{\boldsymbol{x}_{t+1}, \boldsymbol{y}_t\}$ be the vector of all endogenous variables whose value is determined at t (namely, current-period control and next-period state variables). A solution is defined as a set of policy functions for endogenous variables $\boldsymbol{e}_t = \Psi(\boldsymbol{x}_t, \boldsymbol{z}_t)$ that satisfy the optimality conditions (5) and (6) in the relevant region of the state space. Note that the total number of policy functions, $n \equiv n_x + n_y$, is equal to the total number of equations in the system (5) and (6), that is $n_H + n_G$. We assume that functions H and G satisfy jointly a set of regularity conditions that are sufficient to ensure that a steady state exists and is unique and that a solution exists and is unique.

3.2 Construction of hybrid solutions

We develop a method that solves for some policy functions locally (using standard perturbation) and solves for the other policy functions globally (using closed-form expressions and a numerical solver). The method is as follows:

- Step 1. Compute a standard perturbation solution $\widehat{\Psi}$ and partition n policy functions into two parts, $\widehat{\Psi} \equiv \left\{\widehat{\Psi}_1, \widehat{\Psi}_2\right\}$. The perturbation policy functions $\widehat{\Psi}_1$ will be used for constructing a hybrid solution (let n_1 be the total number of such policy functions, $0 \leq n_1 \leq n$). The remaining perturbation policy functions $\widehat{\Psi}_2$ will be replaced with a global solution (the number of such policy functions is $n_2 \equiv n - n_1$).

- Step 2. Partition the system of n equations (5) and (6) into two sub-systems with n_1 and n_2 equations. We assume that the second sub-system with n_2 equations is non-degenerate and identifies n_2 policy functions Ψ_2 uniquely if Ψ_1 is given.

- Step 3. Given the perturbation policy functions $\widehat{\Psi}_1$ chosen in Step 1, solve (either analytically or using a numerical solver) for n_2 policy functions $\widetilde{\Psi}_2 = \Psi_2\left(\widehat{\Psi}_1\right)$ that satisfy n_2 equations chosen in Step 2 for all $(\boldsymbol{x}_t, \boldsymbol{z}_t)$.

The hybrid solution is $\widetilde{\Psi} \equiv \left\{\widehat{\Psi}_1, \widetilde{\Psi}_2\right\}$.

[7] Alternatively, $\boldsymbol{x}_t$ and $\boldsymbol{z}_t$ can be called, respectively, endogenous and exogenous predetermined variables at the beginning of period t, and $\boldsymbol{y}_t$ can be called endogenous non-predetermined variables.

[8] Note that $\boldsymbol{x}_{t+1}$ is t-measurable and therefore, there is no conditional expectation in equations of type (6).

3.3 Properties of hybrid solutions

The properties of hybrid solutions documented for the example in Section 3 carry over to the general case. First, there are multiple hybrid solutions. Namely, in Step 1, there are many ways of choosing how many and which perturbation policy functions to keep, and in Step 2, there are many ways of choosing optimality conditions that identify the remaining policy functions. In particular, two obvious limiting cases of the hybrid method are to keep the original perturbation solution unchanged ($n_1 = n$) and to re-compute the perturbation solution entirely ($n_1 = 0$).Furthermore, to keep the cost of hybrid solutions low, we must choose those optimality conditions for identifying $\widetilde{\Psi}_2$ that are the least expensive to solve (typically, it is easier to solve equations of type (6) than equations of type (5) as the former type does not require the evaluation of conditional expectations). Finally, to make hybrid solutions accurate, we must collect into $\widehat{\Psi}_1$ those policy functions that are computed most accurately by the standard perturbation method (small (large) errors in $\widehat{\Psi}_1$ induce small (large) errors in the hybrid solution).

3.4 Comparison to the literature

Our general presentation of the hybrid method encompasses some previous examples in the literature. First, Dotsey and Mao (1992) compare the performance of a standard linearization method with a method that uses a linearized policy function for capital and labor and solves for consumption and investment allocations analytically from the optimality conditions. In the context of a RBC model with labor and production taxes, none of the two methods strictly dominates the other: the former method produces more accurate policy functions for consumption and real interest rates while the latter method produces more accurate policy functions for investment.

Furthermore, Maliar et al. (2011) extend the hybrid method to include those cases in which optimality conditions do not admit closed-form solutions. They combine a linearized policy functions (used to find capital) with a fixed-point "iteration-on-allocation" numerical solver (used to solve for consumption and labor allocations). In the context of a multi-country RBC model, this modification of the standard linearization method leads to a considerable accuracy improvement. Maliar et al. (2011) also show that the hybrid approach can be also used to improve the performance of global solution methods whose policy functions are not sufficiently accurate.

Finally, pruning methods are another type of perturbation-based methods in the literature that can be classified as hybrid methods. Such methods try to address another potential shortcoming of the standard perturbation methods, namely, their numerical instability in simulation. The term "pruning" was introduced by Kim et al. (2008) who showed that a simulation of high-order perturbation solutions may produce explosive time series. To restore the numerical stability, they propose to replace cross-products of variables in the second-order perturbation solution with cross-products of variables obtained from the first-order perturbation solution. Other papers that focus on stabilizing perturbation methods are Lombardo (2010), and Den Haan and De Wind (forthcoming). In particular, the latter paper uses a fixed-point iteration technique that is similar in spirit to "iteration-on-allocation" solver used for constructing hybrid solutions in the next section of the present paper.

4 Hybrid solutions for a large-scale RBC model

A special 2011's issue of the Journal of Economic Dynamic and Control compares the performance of six state-of-art numerical methods using a collection of multi-country RBC models;

see Kollmann et al. (2011b) for the comparison results. We assess the performance of the hybrid method in the context of a multi-country model used in the above comparison analysis.

4.1 The model

In the description of the model, we follow Juillard and Villemot (2011). The world economy consists of a finite number of countries N, and each country is populated by a representative consumer. A social planner maximizes a weighted sum of expected lifetime utility of the countries' representative consumers

$$\max_{\left\{c_t^j,\ell_t^j,i_t^j,k_{t+1}^j\right\}_{t=0,\ldots,\infty}^{j=1,\ldots,N}} E_0 \sum_{j=1}^{N} \tau^j \left(\sum_{t=0}^{\infty} \beta^t u^j \left(c_t^j, \ell_t^j \right) \right) \tag{8}$$

subject to the world resource constraint,

$$\sum_{j=1}^{N} \left(c_t^j + i_t^j - \delta k_t^i \right) = \sum_{j=1}^{N} \left(a_t^j f^j \left(k_t^j, \ell_t^j \right) - \frac{\phi}{2} k_t^j \left(\frac{i_t^j}{k_t^j} - \delta \right)^2 \right), \tag{9}$$

and to the capital-accumulation equations for $j = 1, \ldots, N$,

$$k_{t+1}^j = (1-\delta)\, k_t^j + i_t^j, \tag{10}$$

where c_t^j, ℓ_t^j, i_t^j, k_t^j, a_t^j, u^j, f^j and τ^j are, respectively, consumption, labor, investment, capital, productivity level, utility function, net production function and welfare weight of a country $j = 1, \ldots, N$; ϕ is the adjustment-cost parameter. Initial condition $(\boldsymbol{k}_0, \boldsymbol{a}_0)$ is given, where $\boldsymbol{k}_t \equiv \left(k_t^1, \ldots, k_t^N\right)$ and $\boldsymbol{a}_t \equiv \left(a_t^1, \ldots, a_t^N\right)$. The process for the productivity level in country j is given by

$$\ln a_t^j = \rho \ln a_{t-1}^j + \sigma \varepsilon_t^j, \tag{11}$$

where $\varepsilon_t^j \equiv e_t + e_t^j$ with e_t and e_t^j being common and country-specific productivity shocks, respectively, and $e_t, e_t^j \sim \mathcal{N}(0,1)$; ρ is the auto-correlation coefficient of the productivity level; and σ determines the standard deviation of the productivity level.

An interior solution to the social planner's problem (8)–(11) satisfies the following optimality conditions for countries $j = 1, \ldots, N$

$$\tau^j u_c^j \left(c_t^j, \ell_t^j \right) = \lambda_t, \tag{12}$$

$$\tau^j u_\ell^j \left(c_t^j, \ell_t^j \right) = -\lambda_t a_t^j f_\ell^j \left(k_t^j, \ell_t^j \right), \tag{13}$$

$$\lambda_t \left[1 + \phi \left(\frac{i_t^j}{k_t^j} - \delta \right) \right] = \beta E_t \left\{ \lambda_{t+1} \left[1 + a_{t+1}^j f_k^j \left(k_{t+1}^j, \ell_{t+1}^j \right) + \phi \left(1 - \delta + \frac{i_{t+1}^j}{k_{t+1}^j} - \frac{1}{2} \left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right) \left(\frac{i_{t+1}^j}{k_{t+1}^j} - \delta \right) \right] \right\}, \tag{14}$$

where λ_t is the Lagrange multiplier associated with world resource constraint (9). Here, and for the remainder of the paper, notation of type F_{χ_m} stands for the first-order partial derivative of a function $F(\ldots, \chi_m, \ldots)$ with respect to a variable χ_m. In addition, the social planner's solution satisfies the set of transversality conditions $\lim_{t\to\infty} \lambda_t k_{t+1}^j = 0$ for $j = 1, \ldots, N$.

4.2 A specific hybrid solution

To solve the model (8)–(11), we construct a specific hybrid solution (in terms of our example of section 2, this hybrid solution corresponds to HYB1).

- The original perturbation solution consists of $4N+1$ policy functions $k_{t+1}^j = \widehat{K}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $c_t^j = \widehat{C}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $\ell_t^j = \widehat{L}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$ and $i_t^j = \widehat{I}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $j = 1, \ldots, N$ and $\lambda_t = \widehat{\Lambda}(\boldsymbol{k}_t, \boldsymbol{a}_t)$ computed using the $4N+1$ optimality conditions (9), (10), (12)–(14).
- To construct the hybrid solution, we take N perturbation policy functions for the next-period capital $k_{t+1}^j = \widehat{K}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$ and compute the remaining $3N+1$ policy functions $c_t^j = \widetilde{C}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $\ell_t^j = \widetilde{L}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $i_t^j = \widetilde{I}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $j = 1, \ldots, N$ and $\lambda_t = \widetilde{\Lambda}(\boldsymbol{k}_t, \boldsymbol{a}_t)$ to satisfy the intra-temporal-choice conditions (9), (10), (12) and (13) (thus, N Euler equations (14) are not used for constructing this specific hybrid solution).

The above construction is motivated by the following accuracy and cost considerations in the context of the given model. First, Maliar et al. (2011) show that one must approximate more accurately the consumption and labor policy functions than the capital policy functions to attain a target accuracy level. Second, Kollmann et al. (2011b) report that the perturbation methods produce smaller errors in inter-temporal-choice conditions (Euler equations) (14) than in the intra-temporal-choice conditions (9), (10), (12) and (13). Third, Maliar et al. (2011) argue that fixing the capital policy functions helps save on cost because one can first simulate the path for the state variables $\{\boldsymbol{k}_{t+1}, \boldsymbol{a}_t\}_{t=1,\ldots,T}$ without solving for the remaining variables $\{\boldsymbol{c}_t, \boldsymbol{\ell}_t, \boldsymbol{i}_t, \lambda_t\}_{t=1,\ldots,T}$ at each date and later fill in such remaining variables, where $\boldsymbol{c}_t \equiv (c_t^1, \ldots, c_t^N)$, $\boldsymbol{\ell}_t \equiv (\ell_t^1, \ldots, \ell_t^N)$, $\boldsymbol{i}_t \equiv (i_t^1, \ldots, i_t^N)$. Finally, disregarding Euler equations (14) helps save on cost because it allows us to avoid expensive multi-dimensional integration.

An example with a closed-form solution for control variables. We illustrate the construction of the hybrid solution using a version of the model (8)–(11) in which control variables can be characterized analytically in terms of state variables. We specifically assume that leisure is non-valued and that the utility function is logarithmic, *i.e.*, $u^j(c_t^j, \ell_t^j) = \ln c_t^j$. This setup corresponds to Model I in Juillard and Villemot (2011).

We take the capital policy functions $k_{t+1}^j = \widehat{K}^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $j = 1, \ldots, N$ delivered by the standard perturbation method and compute the remaining policy functions to satisfy (9), (10) and (12). To be specific, we first find the investment policy function $\widetilde{I}_t^j(\boldsymbol{k}_t, \boldsymbol{a}_t) = i_t^j$, $j = 1, \ldots, N$,

$$i_t^j = \widehat{K}^j(\boldsymbol{k}_t, \boldsymbol{a}_t) - (1-\delta) k_t^j. \tag{15}$$

Given $\widetilde{I}_t^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, we compute aggregate consumption $c_t \equiv \sum_{j=1}^N c_t^j$,

$$c_t = \sum_{j=1}^N \left(a_t^j f^j\left(k_t^j, 1\right) - \frac{\phi}{2} k_t^j \left(\frac{\widetilde{I}_t^j(\boldsymbol{k}_t, \boldsymbol{a}_t)}{k_t^j} - \delta \right)^2 - \widetilde{I}_t^j(\boldsymbol{k}_t, \boldsymbol{a}_t) + \delta k_t^j \right). \tag{16}$$

Condition (12) implies $\frac{\tau^j}{c_t^j} = \lambda_t$, and hence aggregate consumption is $c_t = \sum_{j=1}^N \frac{\tau^j}{\lambda_t}$. The latter condition implies the policy function for the Lagrange multiplier, $\widetilde{\Lambda}(\boldsymbol{k}_t, \boldsymbol{a}_t) = \lambda_t$,

$$\lambda_t = \frac{1}{c_t} \sum_{j=1}^N \tau^j. \tag{17}$$

Finally, the policy functions for individual consumption, $\widetilde{C}_t^j(\boldsymbol{k}_t, \boldsymbol{a}_t) = c_t^j$, $j = 1, \ldots, N$, are

$$c_t^j = \frac{\tau^j}{\sum_{j=1}^N \tau^j} c_t. \tag{18}$$

By construction, policy functions $\widetilde{C}_t^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $\widetilde{I}_t^j(\boldsymbol{k}_t, \boldsymbol{a}_t)$, $j = 1, \ldots, N$ and $\lambda_t = \widetilde{\Lambda}(\boldsymbol{k}_t, \boldsymbol{a}_t)$ satisfy optimality conditions (9), (10) and (12) exactly (*i.e.*, approximation errors in these conditions are zeros).[9]

Solving for control variables numerically. In general, the system of intra-temporal-choice conditions (9), (10), (12) and (13) does not admit a closed-form solution. To construct the policy functions for the control variables numerically, we can use a standard Newton-type numerical solver, though the cost of such a solver can be prohibitively high, especially in large-scale applications. Maliar et al. (2011) develop two alternative methods that are tractable in the given context even if the dimensionality of the problem is high: one is a derivative-free "iteration-on-allocation" numerical solver, and the other is a method of precomputing the intra-temporal-choice functions outside the main iterative cycle.

4.3 Numerical experiments

We apply our hybrid method to solve one of the RBC models studied in the comparison analysis of Kollmann et al. (2011b), namely, Model II with an asymmetric specification. We chose this model because it represents all challenges posed in the comparison analysis: a large number of state variables, endogenous labor-leisure choice, heterogeneity in fundamentals and the absence of closed-form expressions for next-period state and control variables.[10]

4.3.1 Implementation details

In this section, we describe the parametrization of the model, the solution procedure, the accuracy tests implemented and the software and hardware used.

Parametrization of the model. The utility and production functions are given by

$$u^j\left(c_t^j, \ell_t^j\right) = \frac{\left(c_t^j\right)^{1-1/\gamma^j}}{1-1/\gamma^j} - b_j \frac{\left(c_t^j\right)^{1+1/\eta^j}}{1+1/\eta^j}, \qquad f^j\left(k_t^j, \ell_t^j\right) = A\left(k_t^j\right)^{\alpha}\left(\ell_t^j\right)^{1-\alpha},$$

where $\{\gamma^j, b_j, \eta^j\}$ are the utility-function parameters, α is the capital share in production, and A is the normalizing constant in output. To make our results comparable to Kollmann et al. (2011b), we calibrate the model as is done in Juillard and Villemot (2011). We use the following values of common-for-all-countries parameters: $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\sigma = 0.01$, $\rho = 0.95$, $\phi = 0.5$, and we assume that the country-specific utility-function parameters γ^j and η^j are uniformly distributed in the intervals $[0.25, 1]$ and $[0.1, 1]$ across countries $j = 1, \ldots, N$, respectively. The steady state level for productivity is normalized to one, $\overline{a}^j = 1$. We also normalize the steady state levels of capital and labor to one, $\overline{k}^j = 1$, $\overline{\ell}^j = 1$, which implies $\overline{c}^j = A$, $\overline{i}^j = \delta$, $\overline{\lambda} = 1$ and leads to $A = \frac{1-\beta}{\alpha\beta}$, $\tau^j = u_c^j(A, 1)$ and $b^j = (1-\alpha)A^{1-1/\gamma^j}$.

[9] Maliar and Maliar (2003) show how to use analytical aggregation to derive similar policy functions for control variables in certain classes of heterogeneous-agent models with endogenous labor-leisure choice and agent-specific correlated shocks.

[10] Model I has a degenerate labor-leisure choice, and Models III and IV are identical to Model II up to specific assumptions about preferences and technologies. In the comparison analysis of Kollmann et al. (2011b), the implications about accuracy and speed of the studied methods are similar across Models II–IV.

Solution procedure. We use the Dynare implementation of the standard perturbation method (PER). Dynare is a software platform for handling (*i.e.*, solving, simulating, estimating) a wide class of economic models, and in particular, RBC and dynamic stochastic general equilibrium models based on the rational expectations paradigm. Dynare is able to deliver standard perturbation solutions up to third order.[11]

To implement our hybrid method (HYB), we solve for the values of the control variables satisfying the system of the intra-temporal-choice conditions. To be more specific, we compute a sequence for investment $\{\boldsymbol{i}_t\}_{t=1,\ldots,T}$ using (10), and we find $2N+1$ allocations $\{\boldsymbol{c}_t, \boldsymbol{\ell}_t, \lambda_t\}_{t=1,\ldots,T}$ satisfying (9), (12) and (13) using the iteration-on-allocation numerical solver with a target unit-free accuracy of 10^{-10} and damping parameter $\varsigma = 0.01$; see Maliar et al. (2011) for further details on how this solver is implemented in the context of the model (8)–(11).

Accuracy tests. In the implementation of the accuracy tests, we follow Juillard and Villemot (2011). This makes our accuracy results directly comparable to those in Kollmann et al. (2011b). Namely, we compute the size of approximation errors in the $4N+1$ optimality conditions, world resource constraint (9) and optimality conditions (12), (13) and (14) for $j = 1, \ldots, N$. We evaluate errors on two different kinds of domain: one is a sphere of radius r from the steady state (we consider $r = \{0.01, 0.10, 0.30\}$) and the other is a stochastic simulation of 10,200 observation; these types of accuracy tests are introduced in Judd (1992) and Jin and Judd (2002), respectively.

Hardware and software. We use a workstation with two quad-core Intel® Xeon X5460 processors (clocked at 3.16Ghz), 8Gb of RAM, and 64-bit Debian GNU/Linux.[12] The programs are written in C++, rely on the GNU Scientific Library[13] for mathematical routines and are publicly available.[14]

4.3.2 Numerical results

In this section, we assess the performance of the hybrid method in which the capital policy functions are computed by perturbation and the remaining policy functions are computed using the iteration-on-allocation numerical solver. We compare the accuracy of this hybrid method to that of the standard perturbation method.[15] For both methods, we compute perturbations of first, second and third orders. For comparison, we also provide the accuracy frontier attained in the comparison analysis of Kollmann et al. (2011b) (with a reference to the method that delivers the best accuracy in each case).[16] Note that perturbations of the third order are significantly less computationally costly than second-degree approximations for global projection methods.

In Table 1, we provide the results for the model with $N = 2$. As is seen from the table, PER produces relatively small approximation errors in the Euler equations but larger approximation errors in the intra-temporal-choice conditions (9), (12) and (13) (although errors in the world resource constraint are smaller than those in the other two conditions). In turn, HYB produces

[11] See `http://www.dynare.org` and Adjemian et al. (2011) for more details on Dynare.

[12] See `http://www.debian.org`.

[13] See `http://www.gnu.org/software/gsl/` and Galassi et al. (2003).

[14] See Sébastien Villemot's web site: `http://www.dynare.org/sebastien/`.

[15] Our implementation of the perturbation method differs from that of Kollmann et al. (2011a) in that they first take the logarithms of variables before constructing a Taylor expansion, while we work with the variables in levels.

[16] Regarding degrees of approximation in Kollmann et al. (2011a), Maliar et al. (2011) and Pichler (2011) use second-degree polynomials, while Malin et al. (2011) use Smolyak's polynomial that have four times more terms than a second-degree complete polynomial.

negligible approximation errors in the intra-temporal-choice conditions by construction. The only source of errors for HYB is approximation errors in the Euler equations. In fact, Euler-equation errors for HYB are larger than those for PER. However, such errors are still smaller than errors in the intra-temporal-choice optimality conditions for PER (the exceptions here are the case of first-order perturbation for the test on a sphere with a small radius $r = 0.01$, and second-order perturbation on a sphere with a medium radius $r = 0.1$). As a consequence, HYB dominates PER in accuracy. The accuracy improvements are more pronounced for the test on a stochastic simulation than for the test on a sphere. For example, under the third-order approximation, the overall accuracy increases from 0.107% for PER to 0.002% for HYB (equivalently, by 1.5 $\log_{10}$ units).

The comparison with the accuracy frontier attained in the comparison analysis of Kollmann et al. (2011b) shows the following tendencies. First, for the test on a sphere with a small radius, $r = 0.01$, second-order HYB is almost as accurate as (second-order) CGA, and third-order HYB is more accurate by 1.58 $\log_{10}$ units than (second-order) CGA. For the test on a sphere with a larger radius of $r = 0.1$, second-order HYB is not more accurate than Smolyak's algorithm, but third-order HYB is (recall that Smolyak's polynomial has four times more terms than a second-degree complete polynomial). For the test on a sphere with the largest radius considered, $r = 0.3$, HYB performs worse than Smolyak's algorithm which is not surprising given that the latter method is a global solution method. Finally, for the test on a stochastic simulation, third-order HYB performs better than CGA, but second-order HYB does not.

To check the robustness of our analysis, in Table 2, we report the corresponding results for the model with $N = 8$. Overall, the tendencies are very similar to those observed in Table 1. The differences in accuracy between PER and HYB are even more significant reaching almost two orders of magnitude in the test on a stochastic simulation. For the test on a sphere with a small radius, $r = 0.01$, second-order approximations produced by HYB are significantly more accurate than those produced by all other methods compared in Kollmann et al. (2011b). For all the other tests, third-order HYB significantly outperforms the global methods compared in Kollmann et al. (2011b), while second-order HYB is still not better than these global methods.

As far as the cost is concerned, the solution time reported in Tables 1 and 2 for HYB coincides with that for PER. This reflects the fact that HYB uses the perturbation solution for capital and does not construct an explicit solution for the other variables. Such a solution is obtained only in the testing procedure. Furthermore, the tables indicate that HYB requires more time to run the accuracy tests than PER. Specifically, for $N = 2$, the differences in running time between HYB and PER are larger for $N = 2$ than for $N = 8$. However, HYB is still much less expensive than any global method in the literature. The relatively high cost of hybrid solutions is largely explained by a specific implementation of the iteration-on-allocation solver, namely, by using this solver in the point-by-point manner. As mentioned above, using a vectorized version of this solver will reduce the cost of hybrid solutions dramatically.

In Table 3, we report the unit-free maximum absolute differences between the simulated time series produced by any pair of the solution methods considered. In addition to the pairwise comparisons between the first-, second- and third-order PER and HYB, we make a comparison with CGA, which presumably delivers very accurate solutions. We consider the model with $N = 2$, and present the results for consumption, labor, investment and capital of countries 1 and 2. First of all notice that as the order of approximation for PER and HYB increases, the difference between time-series solutions produced by these methods and CGA monotonically decreases. An exception is an HYB solution for capital of countries 1 and 2 in which case the difference between CGA and HYB is the smallest for second-order HYB. Furthermore, in most cases the differences between HYB and CGA solutions are smaller than between PER and CGA solutions, the sole exception being consumption for which third-order PER is closer to CGA

than third-order HYB.

5 Conclusion

The comparison analysis of Kollmann et al. (2011b) documented a trade-off between accuracy and speed of the studied solution methods. We are not subject to such a tradeoff. Our simple hybrid method is both faster and more accurate than any global solution method participating in the above comparison. We attain higher accuracy in particular because we are able to compute third-degree polynomial approximations while the methods studied in Kollmann et al. (2011b) are limited to second-degree polynomials. The cost of global solution methods can be reduced, for example, by using less expensive integration rules and by precomputing integrals outside the main iterative cycle; see Maliar et al. (2011) and Judd et al. (2011a,b) for a discussion. However, even if global third-degree polynomial approximations were feasible, they would be orders of magnitude more expensive than our simple hybrid solutions. Our current objective is to incorporate a routine constructing hybrid solutions in automated perturbation software such as Dynare: this routine must compute different hybrid solutions, compare their accuracy and deliver the most accurate solution to the user.

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Table 1: Multi-country model with $N = 2$ countries: maximum absolute errors by equation on a sphere and on a stochastic simulation.

	1st order		2nd order		3rd order		Accuracy frontier in
	PER	HYB	PER	HYB	PER	HYB	Kollmann et al. (2011b)
Solution time	0.01	0.01	0.02	0.02	0.04	0.04	1602.1 (CGA) 2.2 (SMOL)
Radius $r = 0.01$							
EulerEq	−5.91	−3.83	−6.87	−5.95	−8.41	−7.60	
MUCons	−4.39	–	−5.26	–	−6.69	–	
MULabor	−4.32	–	−5.40	–	−7.03	–	
WorResConst	−4.64	–	−6.09	–	−7.83	–	
Overall	−4.32	−3.83	−5.26	−5.95	−6.69	−7.60	−6.02 (CGA)
Time to run test	0.10	4.51	0.14	3.29	0.27	2.53	
Radius $r = 0.10$							
EulerEq	−4.07	−2.44	−4.12	−3.63	−5.49	−5.20	
MUCons	−2.39	–	−3.69	–	−4.76	–	
MULabor	−2.25	–	−3.64	–	−4.81	–	
WorResConst	−2.63	–	−3.88	–	−5.20	–	
Overall	−2.25	−2.44	−3.64	−3.63	−4.76	−5.20	−4.40 (SMOL)
Time to run test	0.10	5.02	0.14	4.13	0.27	3.42	
Radius $r = 0.30$							
EulerEq	−3.08	−1.38	−2.62	−2.03	−3.55	−3.06	
MUCons	−1.43	–	−2.16	–	−2.84	–	
MULabor	−1.11	–	−1.99	–	−2.87	–	
WorResConst	−1.52	–	−2.25	–	−3.16	–	
Overall	−1.11	−1.38	−1.99	−2.03	−2.84	−3.06	−3.29 (SMOL)
Time to run test	0.10	5.62	0.14	5.07	0.27	4.67	
Stochastic simulation							
EulerEq	−3.41	−2.22	−3.37	−3.44	−4.02	−4.69	
MUCons	−1.55	–	−2.25	–	−2.97	–	
MULabor	−1.57	–	−2.29	–	−3.03	–	
WorResConst	−2.10	–	−3.18	–	−4.11	–	
Overall	−1.55	−2.22	−2.25	−3.44	−2.97	−4.69	−4.50 (CGA)
Time to run test	0.98	49.48	1.42	40.43	2.67	33.03	

Notes: For each model equation (listed in the 1st column), the table reports maximum absolute errors in $\log_{10}$ units across countries and test points. For panels "Radius $r = 0.01$", "Radius $r = 0.10$" and "Radius $r = 0.30$", the set of test points is 1,000 draws of state variables located on spheres with radii 0.01, 0.10 and 0.30, respectively; for panel "stochastic simulation", the set of test points is a stochastic simulation of 10,000 periods. An entry "–" is used if the accuracy measure is below −10 (such errors are viewed as negligible).

Model equations are as follows: "EulerEq" is Euler equation (14); "MUCons" equates the (scaled) marginal utility of consumption to the Lagrange multiplier, see (12); "MULabor" equates the (scaled) marginal utility of labor to marginal productivity of labor multiplied by the Lagrange multiplier, see (13); "WorResConst" is world resource constraint (9); "Overall" is the maximum error across all the model's equations; "Solution time" is time for computing a solution, and "Time to run test" is time for running a test in seconds; and "CGA" and "SMOL" are abbreviations for the cluster-grid, and Smolyak's methods, respectively.

Table 2: Multi-country model with $N = 8$ countries: maximum absolute errors by equation on a sphere and on a stochastic simulation.

	1st order		2nd order		3rd order		Accuracy frontier in
	PER	HYB	PER	HYB	PER	HYB	Kollmann et al. (2011b)
Solution time	0.03	0.03	0.06	0.06	1.32	1.32	27785.8 (CGA) 12126.8 (SMOL)
Radius $r = 0.01$							
EulerEq	−6.26	−3.85	−7.07	−6.36	−7.85	−7.37	
MUCons	−5.28	–	−5.41	–	−6.23	–	
MULabor	−4.60	–	−5.34	–	−6.57	–	
WorResConst	−5.25	–	−6.22	–	−7.65	–	
Overall	−4.60	−3.85	−5.34	−6.36	−6.23	−7.37	−6.03 (CGA)
Time to run test	0.86	14.75	3.29	13.77	26.73	36.19	
Radius $r = 0.10$							
EulerEq	−4.20	−3.23	−5.04	−4.63	−6.76	−6.11	
MUCons	−3.28	–	−4.54	–	−6.02	–	
MULabor	−2.55	–	−3.93	–	−5.48	–	
WorResConst	−3.25	–	−4.75	–	−6.18	–	
Overall	−2.55	−3.23	−3.93	−4.63	−5.48	−6.11	−4.96 (CGA)
Time to run test	0.85	15.37	3.29	15.19	27.00	36.59	
Radius $r = 0.30$							
EulerEq	−3.17	−2.14	−3.57	−3.06	−4.68	−4.22	
MUCons	−2.32	–	−3.77	–	−4.74	–	
MULabor	−1.47	–	−2.46	–	−3.51	–	
WorResConst	−2.29	–	−3.38	–	−4.29	–	
Overall	−1.47	−2.14	−2.46	−3.06	−3.51	−4.22	−4.07 (SMOL)
Time to run test	0.85	17.37	3.36	17.93	26.97	39.46	
Stochastic simulation							
EulerEq	−3.53	−2.30	−3.49	−3.45	−4.37	−5.02	
MUCons	−1.79	–	−2.66	–	−3.52	–	
MULabor	−1.58	–	−2.40	–	−3.22	–	
WorResConst	−2.22	–	−3.44	–	−4.48	–	
Overall	−1.58	−2.30	−2.40	−3.45	−3.22	−5.02	−4.54 (CGA)
Time to run test	8.48	157.77	34.48	160.56	269.86	373.92	

Notes: For each model equation (listed in the 1st column), the table reports maximum absolute errors in $\log_{10}$ units across countries and test points. For panels "Radius $r = 0.01$", "Radius $r = 0.10$" and "Radius $r = 0.30$", the set of test points is 1,000 draws of state variables located on spheres with radii 0.01, 0.10 and 0.30, respectively; for panel "stochastic simulation", the set of test points is a stochastic simulation of 10,000 periods. An entry "–" is used if the accuracy measure is below −10 (such errors are viewed as negligible).

Model equations are as follows: "EulerEq" is Euler equation (14); "MUCons" equates the (scaled) marginal utility of consumption to the Lagrange multiplier, see (12); "MULabor" equates the (scaled) marginal utility of labor to marginal productivity of labor multiplied by the Lagrange multiplier, see (13); "WorResConst" is world resource constraint (9); "Overall" is the maximum error across all the model's equations; "Solution time" is time for computing a solution, and "Time to run test" is time for running a test in seconds; and "CGA" and "SMOL" are abbreviations for the cluster-grid, and Smolyak's methods, respectively.

Table 3: Multi-country model with $N = 2$ countries: maximum differences across simulated series.

	1st order		2nd order		3rd order	
Variable	PER	HYB	PER	HYB	PER	HYB
Consumption of country 1						
1st order HYB	−2.40					
2nd order PER	−2.54	−2.63				
2nd order HYB	−2.53	−2.66	−3.44			
3rd order PER	−2.54	−2.65	−3.42	−3.91		
3rd order HYB	−2.54	−2.65	−3.46	−3.91	−4.44	
CGA	−2.54	−2.64	−3.45	−4.02	−4.21	−4.17
Consumption of country 2						
1st order HYB	−1.74					
2nd order PER	−1.88	−2.04				
2nd order HYB	−1.88	−2.06	−3.09			
3rd order PER	−1.89	−2.05	−3.32	−3.31		
3rd order HYB	−1.89	−2.05	−3.35	−3.31	−4.34	
CGA	−1.88	−2.04	−3.19	−3.42	−3.65	−3.57
Labor supply of country 1						
1st order HYB	−2.75					
2nd order PER	−3.09	−3.01				
2nd order HYB	−3.08	−3.02	−3.58			
3rd order PER	−3.07	−3.02	−3.57	−4.36		
3rd order HYB	−3.07	−3.02	−3.62	−4.36	−4.53	
CGA	−3.08	−3.02	−3.62	−4.46	−4.48	−4.66
Labor supply of country 2						
1st order HYB	−2.15					
2nd order PER	−2.28	−2.16				
2nd order HYB	−2.27	−2.17	−2.95			
3rd order PER	−2.28	−2.16	−2.88	−3.48		
3rd order HYB	−2.28	−2.16	−2.94	−3.44	−3.71	
CGA	−2.28	−2.15	−2.94	−3.53	−3.65	−3.73
Investment of country 1						
1st order HYB	−1.39					
2nd order PER	−1.42	−1.41				
2nd order HYB	−1.42	−1.42	−2.85			
3rd order PER	−1.41	−1.42	−2.70	−2.73		
3rd order HYB	−1.41	−1.42	−2.70	−2.73	−4.12	
CGA	−1.41	−1.42	−2.69	−2.84	−2.84	−2.85
Investment of country 2						
1st order HYB	−1.33					
2nd order PER	−1.23	−1.50				
2nd order HYB	−1.23	−1.51	−2.87			
3rd order PER	−1.24	−1.51	−2.66	−2.69		
3rd order HYB	−1.24	−1.51	−2.67	−2.69	−4.17	
CGA	−1.23	−1.50	−2.71	−2.80	−2.84	−2.85
Capital of country 1						
1st order HYB	−1.94					
2nd order PER	−1.82	−1.97				
2nd order HYB	−1.81	−1.99	−3.20			
3rd order PER	−1.81	−1.99	−3.18	−3.17		
3rd order HYB	−1.81	−1.99	−3.20	−3.16	−4.51	
CGA	−1.81	−1.99	−3.28	−3.48	−3.45	−3.44
Capital of country 2						
1st order HYB	−1.95					
2nd order PER	−1.81	−2.19				
2nd order HYB	−1.82	−2.22	−3.31			
3rd order PER	−1.82	−2.20	−3.23	−3.54		
3rd order HYB	−1.82	−2.21	−3.24	−3.55	−4.62	
CGA	−1.82	−2.22	−3.30	−3.69	−3.51	−3.53

Notes: For each model variable, the table reports maximum absolute values (in $\log_{10}$ units) of relative differences between simulated series. The comparison is done pairwise across solution methods. "CGA" is an abbreviation for the cluster-grid method.