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Semi-Global Solutions to DSGE Models: Perturbation around a Deterministic Path

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Abstract

This study presents an approach based on a perturbation technique to construct global solutions to dynamic stochastic general equilibrium models (DSGE). The main idea is to expand a solution in a series of powers of a small parameter scaling the uncertainty in the economy around a global solution to the deterministic model, i.e. the model where the volatility of the shocks vanishes. Under the assumption that a deterministic path is already known the higher order terms in the expansion are obtained recursively by solving linear rational expectations models with time-varying parameters. The present work proposes a method based on backward recursion for solving this type of models. The conditions under which the solutions exist are found.

Keywords: DSGE, perturbation, rational expectations, time-varying parameters, backward induction
JEL: C62, D58, D84

1. Introduction

Perturbation methods are the most popular approach to solve nonlinear DSGE models owing to their ability to deal with medium and large-size models for reasonable computational time. Perturbations applied in macroeconomics are used to expand the exact solution around a deterministic steady state in powers of state variables and a small parameter scaling the uncertainty in the economy. The solutions based on the Taylor series expansion

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are intrinsically local, i.e. they are accurate in some neighborhood (presumably small) of the deterministic steady state (Kim et al. (2008); Den Haan, and De Wind (2012)). Out of the neighborhood the solutions may behave odd, for example, can imply explosive dynamics. The other problem with the perturbation method is that we do not know a priori for non-trivial models how small the neighborhood must be to achieve a given level of accuracy.

The recent crisis has renewed an interest in methods that provide global solutions to DSGE models, i.e. the solutions some points of which are far away from the steady state. This may occur after a big shock hitting the economy, or if the initial conditions are far away from the steady state, the examples of this situation are the economies in transition and developing economies.

This study presents an approach based on a perturbation technique to construct global solutions to DSGE models. The proposed solutions are represented as a series in powers of a small parameter σ scaling the covariance matrix of the shocks. The zero order approximation corresponds to the solution to the deterministic model, because all shocks vanish as $\sigma = 0$. Global solutions to deterministic models can be obtained reasonably fast by effective numerical methods[2] even for large size models (Hollinger (2008)). For this reason the next stages of the method are implemented assuming that the solution to the deterministic model under given initial conditions is known.

Higher-order systems depend only on quantities of lower orders, hence they can be solved recursively. The homogeneous part of these systems is the same for all orders and depends on the deterministic solution. Consequently, each system can be represented as a rational expectation model with time-varying parameters. In the case of rational expectations models with constant parameters the stable block of equations can be isolated and solved forward. This is not possible for models with time-varying parameters.

The present work proposes a method for solving this type of models and determines the conditions under which the solutions of the method exist. The method starts with finding a finite-horizon solution by using backward recursion. Next we prove that as the horizon tends to infinity the finite-

[2]The algorithms incorporated in the widely-used software such as Dynare (and less available Troll) find a stacked-time solution and are based on Newton's method combined with sparse-matrix techniques (Adjemian, Bastani, Juillard, Karamé, Mihoubi, Perendial, Pfeifer, Ratto, and Villemot (2011)).

horizon solutions approach to a limit solution that is bounded for all positive time.

Notice that whenever the deterministic solution is global in state variables so is the approximate solution to the stochastic problem. For this reason, we shall call this approach semi-global. In addition, the method can handle the stochastic simulation of models with anticipated structural and policy change.

To illustrating how the method works in practice, we apply it to the asset pricing model of Burnside (1998). The simplicity of the model allows for obtaining the approximations in an analytical form. We compare the policy functions of the second order solution of the semi-global method with the local Taylor series expansion of order two (Schmitt-Grohé, and Uribe (2004)). In contrast to local Taylor series expansion the semi-global method inherits global properties from the exact solution such as monotonicity.

This paper contributes to a growing literature on using the perturbation technique for solving DSGE models. The perturbation methodology in economics has been advanced by Judd and co-authors as in Judd (1998); Gaspar, and Judd (1997); Judd, and Guu (1997). Jin, and Judd (2002) give a theoretical basis for using perturbation methods in DSGE modeling; namely, applying the implicit function theorem, they prove that the perturbed rational expectations solution continuously depends on a parameter and therefore tends to the deterministic solution as the parameter tends to zero.

Almost all of the literature is concerned with the approximations around the steady state as in Collard, and Juillard (2001); Schmitt-Grohé, and Uribe (2004); Kim et al. (2008); Gomme, and Klein (2011). Lombardo (2010) and Lombardo, and Uhlig (2014) make use of series expansion in powers of σ to provide a theoretical foundation for pruning methods (Kim et al. (2008)), which is aimed to avoid the explosive behaviour of a solution. Lombardo and Uhlig's approach can be treated as a special case of the method proposed in this study, namely a deterministic solution around which the expansion is used is only the steady state. However, the solution obtained by the pruning procedure remains local, and as such may provide a first few impulse responses with wrong signs under a sufficiently large shock. This case seems even worse than the explosive dynamics since the impulse responses for a first few periods are most interesting and relevant for theoretical implications of a model as well as a policy analysis; therefore, their incorrect signs could mislead a researcher or a policymaker.

The rest of the paper is organized as follows. The next section presents the model set-up. Section 3 provides a detailed exposition of series expansions for DSGE models. In Section 4 we transform the model into a convenient form to deal with. Section 5 presents the method for solving rational expectations models for time-varying parameters. The proposed method is applied to an asset pricing model in Section 6, where it is also compared with the local Taylor series expansions. Conclusions are presented in Section 7.

2. The Model

DSGE models usually have the form

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t, z_{t+1}, z_t) = 0, \tag{1}$$

$$z_{t+1} = \Lambda z_t + \sigma \varepsilon_{t+1}, \tag{2}$$

where E_t denotes the conditional expectations operator, x_t is an $n_x \times 1$ vector containing the t-period endogenous state variables; y_t is an $n_y \times 1$ vector containing the t-period endogenous variables that are not state variables; z_t is an $n_z \times 1$ vector containing the t-period exogenous state variables; ε_t is the vector with the corresponding innovations; and the $n_z \times n_z$ covariance matrix Ω; f maps $\mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_z}$ into $\mathbb{R}^{n_y} \times \mathbb{R}^{n_x}$ and is assumed to be sufficiently smooth. The scalar σ ($\sigma > 0$) is a scaling parameter for the disturbance terms ε_t. We assume that all mixed moments of ε_t are finite. All eigenvalues of the matrix Λ have modulus less than one.

The problem is to find a stable solution (x_t, y_t) to (1) for a given initial condition (x_0, z_0). A process is stable if its unconditional expectations are bounded (Klein (2000)).

3. Series Expansion

In this section we shall follow the perturbation methodology used in applied mathematics (see, for example, Nayfeh (1973) and Holmes (2013)) to derive an approximate solution to the model (1)–(2). For small σ, we assume that the solution has a particular form of expansions

$$y_t = y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)} + \cdots \tag{3}$$

$$x_t = x_t^{(0)} + \sigma x_t^{(1)} + \sigma^2 x_t^{(2)} + \cdots \tag{4}$$

The exogenous process z_t can also easily be represented in the form of expansion in σ

$$z_t = z_t^{(0)} + \sigma z_t^{(1)}. \tag{5}$$

Indeed, plugging (5) into (2) gives

$$z_{t+1} = z_{t+1}^{(0)} + \sigma z_{t+1}^{(1)} = \Lambda(z_t^{(0)} + \sigma z_t^{(1)}) + \sigma\varepsilon_{t+1}.$$

Collecting the terms of like powers of σ and equating them to zero, we get

$$z_{t+1}^{(0)} = \Lambda z_t^{(0)}, \tag{6}$$

$$z_{t+1}^{(1)} = \Lambda z_t^{(1)} + \varepsilon_{t+1}. \tag{7}$$

Since the expansion (5) must be valid for all σ at the initial time $t = 0$, the initial conditions are

$$z_0^{(0)} = z_0 \quad \text{and} \quad z_0^{(1)} = 0. \tag{8}$$

Then substituting (3), (4) and (5) into (1), we have

$$\begin{aligned} &E_t f(y_{t+1}^{(0)} + \sigma y_{t+1}^{(1)} + \sigma^2 y_{t+1}^{(2)} + \cdots, y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)} + \cdots, \\ &x_{t+1}^{(0)} + \sigma x_{t+1}^{(1)} + \sigma^2 x_{t+1}^{(2)} + \cdots, x_t^{(0)} + \sigma x_t^{(1)} + \sigma^2 x_t^{(2)} + \cdots, \\ &z_{t+1}^{(0)} + \sigma z_{t+1}^{(1)}, z_t^{(0)} + \sigma z_t^{(1)}) = 0. \end{aligned} \tag{9}$$

Expanding the left hand side of (9) for small σ, collecting the terms of like powers of σ and setting their coefficients to zero, we obtain

Coefficient of σ^0

$$f(y_{t+1}^{(0)}, y_t^{(0)}, x_{t+1}^{(0)}, x_t^{(0)}, z_{t+1}^{(0)}, z_t^{(0)}) = 0. \tag{10}$$

The requirement that (4) and (5) must hold for all arbitrary small σ implies that the initial conditions for (10) are

$$z_0^{(0)} = z_0 \quad \text{and} \quad x_0^{(0)} = x_0. \tag{11}$$

The terminal conditions for $y_\infty^{(0)}$ and $x_\infty^{(0)}$ are the deterministic steady states

$$y_\infty^{(0)} = \bar{y} \quad \text{and} \quad x_\infty^{(0)} = \bar{x}. \tag{12}$$

The system of equations (6) and (10) is a deterministic model since it corresponds to the model (1) and (2), where all shocks vanish (for this reason we omit the expectations operator in (10)). The deterministic model (6) and (10) with the initial and terminal conditions (11) and (12), respectively, can be solved globally by a number of effective algorithms, for example the extended path method (Fair, and Taylor (1983)) or a Newton-like method (for example, Juillard (1996)). As this study is primarily concerned with stochastic models, in what follows we suppose that the deterministic model is already solved and its solution is known.

Coefficient of σ^1

$$\begin{aligned}&E_t\{f_{1,t}\cdot y_{t+1}^{(1)}+f_{2,t}\cdot y_t^{(1)}+f_{3,t}x_{t+1}^{(1)}+f_{4,t}x_t^{(1)}\\&+f_{5,t}z_{t+1}^{(1)}+f_{6,t}z_t^{(1)}\}=0.\end{aligned}\tag{13}$$

The matrices

$$f_{i,t}=f_i\left(y_{t+1}^{(0)},y_t^{(0)},x_{t+1}^{(0)},x_t^{(0)},z_{t+1}^{(0)},z_t^{(0)}\right),i=1,\dots,6,$$

are the Jacobian matrices of the mapping f with respect to the ith argument (that is y_{t+1}, y_t, x_{t+1}, x_t, z_{t+1}, and z_t, respectively), at the point $\left(y_{t+1}^{(0)},y_t^{(0)},x_{t+1}^{(0)},x_t^{(0)},z_{t+1}^{(0)},z_t^{(0)}\right)$. The requirement that (4) must hold for all arbitrary small σ implies that the initial condition for (15) is

$$x_0^{(1)}=0.\tag{14}$$

Coefficient of σ^n, $n>1$

$$E_t\Big\{f_{1,t}\cdot y_{t+1}^{(n)}+f_{2,t}\cdot y_t^{(n)}+f_{3,t}\cdot x_{t+1}^{(n)}+f_{4,t}\cdot x_t^{(n)}+\eta_{t+1}^{(n)}\Big\}=0.\tag{15}$$

The requirement that (4) must hold for all arbitrary small σ implies that the initial condition for (15) is

$$x_0^{(n)}=0.\tag{16}$$

A nice feature of the set of systems of equations (15) is that the linear homogeneous part $f_{i,t}$ is the same for all $n>0$. The difference is only in the non-homogeneous terms $E_t\eta_{t+1}^{(n)}$ that are some mappings for which the set of arguments includes only quantities of order less than n

$$\left(y_{t+1}^{(0)},y_t^{(0)},x_{t+1}^{(0)},x_t^{(0)},\dots,y_{t+1}^{(n-1)},y_t^{(n-1)},x_{t+1}^{(n-1)},x_t^{(n-1)},z_{t+1}^{(0)},z_t^{(0)},z_{t+1}^{(1)},z_t^{(1)}\right).$$

Particularly, for $n=1,2$ we have

$$E_t\eta_{t+1}^{(1)}=(f_{5,t}\Lambda+f_{6,t})z_t^{(1)},$$

and

$$
\begin{aligned}
E_t \eta_{t+1}^{(2)} = E_t \{ & \tfrac{1}{2} f_{11,t} \left(y_{t+1}^{(1)} \right)^2 + \tfrac{1}{2} f_{22,t} \left(y_t^{(1)} \right)^2 + \tfrac{1}{2} f_{33,t} \left(x_{t+1}^{(1)} \right)^2 \\
& + \tfrac{1}{2} f_{44,t} \left(x_t^{(1)} \right)^2 + \tfrac{1}{2} f_{55,t} \left(z_{t+1}^{(1)} \right)^2 + \tfrac{1}{2} f_{66,t} \left(z_t^{(1)} \right)^2 + f_{12,t} y_{t+1}^{(1)} y_t^{(1)} \\
& + f_{13,t} y_{t+1}^{(1)} x_{t+1}^{(1)} + f_{14,t} y_{t+1}^{(1)} x_t^{(1)} + f_{15,t} y_{t+1}^{(1)} z_{t+1}^{(1)} + f_{16,t} y_{t+1}^{(1)} z_t^{(1)} \\
& + f_{23,t} y_t^{(1)} x_{t+1}^{(1)} + f_{24,t} y_t^{(1)} x_t^{(1)} + f_{25,t} y_t^{(1)} z_{t+1}^{(1)} + f_{26,t} y_t^{(1)} z_t^{(1)} \\
& + f_{34,t} x_{t+1}^{(1)} x_t^{(1)} + f_{35,t} x_{t+1}^{(1)} z_{t+1}^{(1)} + f_{36,t} x_{t+1}^{(1)} z_t^{(1)} + f_{45,t} x_t^{(1)} z_{t+1}^{(1)} \\
& + f_{46,t} x_t^{(1)} z_t^{(1)} + f_{56,t} z_{t+1}^{(1)} z_t^{(1)} \},
\end{aligned} \tag{17}
$$

respectively; where $f_{ij,t}$, $i = 1, \ldots, 6$, $j = 1, \ldots, 6$, denotes the mixed partial Frechét derivative of f_t of order two with respect to ith and jth arguments at the point

$$\left(y_{t+1}^{(0)}, y_t^{(0)}, x_{t+1}^{(0)}, x_t^{(0)}, z_{t+1}^{(0)}, z_t^{(0)} \right). \tag{18}$$

In other words, $f_{ij,t}$ is a bilinear mapping (see, for example, Abraham, Marsden, and Ratiu (2001, p. 55)) depending on vector (18) (and hence on t).[3] The expectations $E_t \eta_{t+1}^{(n)}$ are bounded if all conditional mixed moments of $z_{t+1}^{(1)}$ are bounded up to order n and the vectors (18) are bounded for all $t \geq 0$.

Equation (15) with the initial conditions (16) is a linear rational expectations model with time-varying parameters and bounded the non-homogeneous terms $E_t \eta_{t+1}^{(n)}$. To solve the problem (15)–(14) is equivalent to finding a bounded solution $(x_t^{(n)}, y_t^{(n)})$ for $t > 0$ under the assumption that the bounded solutions to the problems of all orders less than n are already known. Knowing how to solve these types of model and using the structure of mappings $E_t \eta_{t+1}^{(n)}$, we can find recursively solutions, $y_t^{(n)}, x_t^{(n)}$, to (15) for every order n, starting with $n = 1$. In the next section we transform equation (15) in a more convenient form to deal with.

4. Transformation of the Model

Define the deterministic steady state as vectors $(\bar{y}, \bar{x}, 0)$ such that

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0) = 0. \tag{19}$$

[3] We do not make use tensor notation for brevity.

We can represent $f_{i,t}$ in (15) as $f_{i,t} = f_i + \hat{f}_{i,t}$, $i = 1, \ldots, 6$, where

$$f_i = f_i(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0)$$

are the Jacobian matrices of the mapping f at the steady state with respect to ith argument, and

$$\hat{f}_{i,t} = f_{i,t}(y^{(0)}_{t+1}, y^{(0)}_t, x^{(0)}_{t+1}, x^{(0)}_t, z^{(0)}_{t+1}, z^{(0)}_t) - f_i(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0). \tag{20}$$

Note also that $\hat{f}_{i,t} \to 0$ as $t \to \infty$, because a deterministic solution must tend to the deterministic steady state as t tends to infinity. Consequently, $f_{i,t}$ can be thought of as a perturbation of f_i. As Equations (15) have the same form for all $n > 0$, to shorten notation, further on we omit the superscript (n) when no confusion can arise. Therefore Equations (15) can be written in the vector form

$$\Phi_t E_t \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \Lambda_t \begin{bmatrix} x_t \\ y_t \end{bmatrix} + E_t \eta_{t+1}, \tag{21}$$

where $\Phi_t = \left[f_3 + \hat{f}_{3,t}, f_1 + \hat{f}_{1,t} \right]$ and $\Lambda_t = \left[f_4 + \hat{f}_{4,t}, f_2 + \hat{f}_{2,t} \right]$. We assume that the matrices Φ_t are invertible for all $t \geq 0$. For instance, this assumption always holds in some neighborhood of the steady state if the Jacobian $[f_3, f_1]^{-1}$ at the steady state is invertible.[4]

Pre-multiplying (21) by Φ_t^{-1}, we get

$$E_t \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = L \begin{bmatrix} x_t \\ y_t \end{bmatrix} + M_t \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \Phi_t^{-1} E_t \eta_{t+1}, \tag{22}$$

where $L = [f_3, f_1]^{-1} [f_4, f_2]$ and

$$M_t = \left[f_3 + \hat{f}_{3,t}, f_1 + \hat{f}_{1,t} \right]^{-1} \left[f_4 + \hat{f}_{4,t}, f_2 + \hat{f}_{2,t} \right] - [f_3, f_1]^{-1} [f_4, f_2].$$

Particularly, for $n = 1$ we have

$$E_t \begin{bmatrix} x^{(1)}_{t+1} \\ y^{(1)}_{t+1} \end{bmatrix} = L \begin{bmatrix} x^{(1)}_t \\ y^{(1)}_t \end{bmatrix} + M_t \begin{bmatrix} x^{(1)}_t \\ y^{(1)}_t \end{bmatrix} + \Phi_t^{-1} (f_{5,t} \Lambda + f_{6,t}) z^{(1)}_t. \tag{23}$$

[4] This assumption is made for ease of exposition. If $[f_3, f_1]$ is a singular matrix, then further on we must use a generalized Schur decomposition for which derivations remain valid, but become more complicated.

Notice that $lim_{t\to\infty} M_t = 0$. As in the case of rational expectations models with constant parameters it is convenient to transform (22) using the spectral property of L. Namely, the matrix L is transformed into a block-diagonal one

$$L = ZPZ^{-1}, \tag{24}$$

where

$$P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \tag{25}$$

where A and B are matrices with eigenvalues larger and smaller than one (in modulus), respectively; and Z is an invertible matrix[5]. This can be done, for example, by initially transforming L in a simple Schur form $L = Z_1 L_1 Z_1^{-1}$, where Z_1 is a unitary matrix, L_1 is an upper triangular Schur form with the eigenvalues along the diagonal. We then transform the matrix L_1 in the block-diagonal Schur factorization $L_1 = Z_2 P Z_2^{-1}$, where Z_2 is an invertible matrix and P is block-diagonal and each diagonal block is a quasi upper-triangular Schur matrix[6]. Hence the matrix Z in (24) has the form $Z = Z_1 Z_2$. We also impose the conventional Blanchard-Kan condition (Blanchard, and Kahn (1980)) on the dimension of the unstable subspace, i.e., $dim(B) = n_y$.

After introducing the auxiliary variables

$$[s_t, u_t]' = Z^{-1}[x_t, y_t]' \tag{26}$$

and pre-multiplying (22) by Z^{-1}, we have

$$E_t s_{t+1} = A s_t + Q_{11,t} s_t + Q_{12,t} u_t + \Psi_{1t} E_t \eta_{t+1}, \tag{27}$$

$$E_t u_{t+1} = B u_t + Q_{21,t} s_t + Q_{22,t} u_t + \Psi_{2t} E_t \eta_{t+1}, \tag{28}$$

where $[\Psi_{1,t}, \Psi_{2,t}] = Z \Phi_t^{-1}$ and

$$\begin{bmatrix} Q_{11,t} & Q_{12,t} \\ Q_{21,t} & Q_{22,t} \end{bmatrix} = Z M_t Z^{-1}. \tag{29}$$

Particularly, for $n = 1$, we have

$$E_t s_{t+1}^{(1)} = A s_t^{(1)} + Q_{11,t} s_t^{(1)} + Q_{12,t} u_t^{(1)} + \Pi_{1,t} z_t^{(1)},$$

$$E_t u_{t+1}^{(1)} = B u_t^{(1)} + Q_{21,t} s_t^{(1)} + Q_{22,t} u_t^{(1)} + \Pi_{2,t} z_t^{(1)},$$

[5]A simple Schur triangular factorization is also possible to be employed here, but at the cost of more complicated derivations. The block-diagonal structure of the matrix P simplifies algebra.

[6]The function bdschur of Matlab Control System Toolbox performs this factorization.

where

$$\begin{bmatrix} \Pi_{1,t} \\ \Pi_{2,t} \end{bmatrix} = Z\Phi_t^{-1}(f_{5,t}\Lambda + f_{6,t}).$$

System (27)-(28) is a linear rational expectations model with time-varying parameters, therefore to solve the system we cannot apply the approaches used in the case of models with constant parameters (Blanchard, and Kahn (1980); Anderson and Moor (1985); Uhlig (1999); Klein (2000); Sims (2001), etc.). In Subsection 5.2 we devevop a method for solving this type of models.

5. Solving the Rational Expectations Model with Time-Varying Parameters

5.1. Notation

This subsection introduces some notation that will be necessary further on. By $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. The induced norm for a real matrix D is defined by

$$\|D\| = \sup_{|s|=1} |\ Ds|.$$

The matrix Z in (24) can be chosen in such a way that

$$\|A\| < \alpha + \gamma < 1 \text{ and } \|B^{-1}\| < \beta + \gamma < 1, \tag{30}$$

where α and β are the largest eigenvalues (in modulus) of the matrices A and B^{-1}, respectively, and γ is arbitrarily small. This follows from the same arguments as in Hartmann (1982, §IV 9), where it is done for the Jordan matrix decomposition. Note also that $\|B\|^{-1} < 1$ for sufficiently small γ. Let

$$B_t = B + Q_{22,t}, \quad A_t = A + Q_{11,t}. \tag{31}$$

By definition, put

$$a = \sup_{t=0,1,\dots} \|A_t\|, \quad b = \sup_{t=0,1,\dots} \left\|B_t^{-1}\right\|, \tag{32}$$

$$c = \sup_{t=0,1,\dots} \|Q_{12,t}\|, \quad d = \sup_{t=0,1,\dots} \|Q_{21,t}\|. \tag{33}$$

In the sequel, we assume that all the matrices B_t, $t = 0, 1, \dots$, are invertible. Note that the numbers a, b, c and d depend on the initial conditions

$(x_0^{(0)}, z_0^{(0)})$. From the definitions of A_t, A, B_t, B, $Q_{12,t}$ and $Q_{21,t}$ and the condition $lim_{t\to\infty}(x_t^{(0)}, z_t^{(0)}) = (\bar{x}, 0)$, it follows that

$$\lim_{t\to\infty} c(x_t^{(0)}, z_t^{(0)}) = 0, \quad \lim_{t\to\infty} d(x_t^{(0)}, z_t^{(0)}) = 0, \tag{34}$$

$$\lim_{t\to\infty} a(x_t^{(0)}, z_t^{(0)}) = \|A\| < 1, \quad \lim_{t\to\infty} b(x_t^{(0)}, z_t^{(0)}) = \left\|B^{-1}\right\| < 1.$$

This means that c and d can be arbitrary small and

$$a < 1 \quad \text{and} \quad b < 1 \tag{35}$$

by choosing $(x_0^{(0)}, z_0^{(0)})$ close enough to the steady state.

5.2. Solving the transformed system (27)–(28)

Taking into account notation (31), we can rewrite (27)–(28) in the form

$$E_t s_{t+1} = A_t s_t + Q_{12,t} u_t + \Psi_{1,t} E_t \eta_{t+1}, \tag{36}$$
$$E_t u_{t+1} = B_t u_t + Q_{21,t} s_t + \Psi_{2,t} E_t \eta_{t+1}. \tag{37}$$

In this subsection we construct a bounded solution to (36)–(37) for $t \geq 0$ with an arbitrary initial condition $s_0 \in \mathbb{R}^{n_x}$ and find under which conditions this solution exists. For this purpose, we first start with solving a finite-horizon problem with a fixed terminal condition using backward recursion. Then, we prove the convergence of the obtained finite-horizon solutions to a bounded infinite-horizon one as the terminal time T tends to infinity.

Fix a horizon $T > 0$. At the time T using the invertibility of B_T and solving Equation (37) backward, we can obtain u_T as a linear function of s_T, the terminal condition $E_T u_{T+1}$ and the "exogenous" term $\Psi_{2,T} E_T \eta_{T+1}$

$$u_T = -B_T^{-1} Q_{21,T} s_T - B_T^{-1} \Psi_{2,t} E_T \eta_{T+1} + B_T^{-1} E_T u_{T+1}.$$

Proceeding further with backward recursion, we shall obtain finite-horizon solutions for each $t = 0, 1, 2, \ldots, T$. For doing this we need to define the following recurrent sequence of matrices:

$$K_{T,T-i-1} = L_{T,T-i}^{-1} \left(Q_{21,T-i} + K_{T,T-i} A_{T-i}\right), \quad i = 0, 1, \ldots, T, \tag{38}$$

where

$$L_{T,T-i} = B_{T-i} + K_{T,T-i} Q_{12,T-i}, \tag{39}$$

with the terminal condition $K_{T,T+1} = 0$. In (38) and (39) the first subscript T defines the time horizon, while the second subscript defines all times between 0 and $T+1$. Let $u_{T,T-i}$, $i = 0, 1, \ldots, T$, denote the $(T-i)$-time solution obtained by backward recursion that starts at the time T. The matrices (38) and (39) are needed for constructing approximate solutions by backward recursion.

Proposition 5.1. *Suppose that the sequence of matrices* (38) *and* (39) *exists; then the solution to* (36)–(37) *has the following representation:*

$$u_{T,T-i} = -K_{T,T-i}s_{T-i} + g_{T,i} + \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1}\right) E_{T-i}\left(u_{T+1}\right), \tag{40}$$

where $i = 0, 1, \ldots, T$; *and*

$$g_{T,i} = -\sum_{j=1}^{i+1} \prod_{k=1}^{j} L_{T,T-i+k}^{-1} (\Psi_{2,T-i+j} + K_{T,T-i+j}\Psi_{1,T-i+j}) E_{T-i}\eta_{T-i+j}. \tag{41}$$

For the proof see Appendix A. The sequence of matrices (38) exists if all matrices $L_{T,T-i}$, $i = 0, 1, \ldots, T$, are invertible. For this we need, in addition, some boundedness condition on the matrices $B_{T-i}^{-1}K_{T,T-i+1}Q_{12,T-i}$. From (34) the matrices B_{T-i}^{-1} and $Q_{12,T-i}$ are bounded, hence this condition boils down to the boundedness of matrices $K_{T,T-i+1}$.

Proposition 5.2. *If for* a, b, c *and* d *from* (32)–(33) *the inequality*

$$cd < \frac{1}{4}\left(\frac{1}{b} - a\right)^2 = \left(\frac{1-ab}{2b}\right)^2 \tag{42}$$

holds, then

$$\left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i+1}\| \cdot \|Q_{12,T-i}\| < 1, \quad i = 0, 1, 2, \ldots T. \tag{43}$$

For the proof see Appendix A.

Proposition 5.3. *If the inequality* (43) *holds, then the matrices* $L_{T,T-i}$, $i = 0, 1, 2. \ldots, T$, *are invertible.*

Proof. From (39) and the invertibility of B_{T-i} it follows that

$$L_{T,T-i} = B_{T-i}\left(I + B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i}\right). \tag{44}$$

The matrices $L_{T,T-i}$ are invertible if and only if the matrices $\left(I + B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i}\right)$ are invertible. From the norm property and (43) we have

$$\left\|B_{T-i}^{-1} K_{T,T-i+1} Q_{12,T-i}\right\| \leq \left\|B_{T-i}^{-1}\right\| \cdot \left\|K_{T,T-i+1}\right\| \cdot \left\|Q_{12,T-i}\right\| < 1.$$

The invertibility of $\left(I + B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i}\right)$ now follows from Golub, and Van Loan (1996, Lemma 2.3.3). □

For $i = T$ from (40) we have

$$u_{T,0} = -K_{T,0} s_0 + g_{T,T} + \left(\prod_{k=1}^{T+1} L_{T,k}^{-1}\right) E_0\left(u_{T+1}\right). \tag{45}$$

This is a finite-horizon solution to the rational expectations model with time-varying coefficients (36)–(37) and with a given initial condition s_0. What is left is to show that the solution $u_{T,0}$ of the form (45) converges to some limit as $T \to \infty$.

Proposition 5.4. *If inequality (42) holds, then the limit*

$$\lim_{T\to\infty} K_{T,j} = K_{\infty,j} \quad \textit{for} \quad j = 0, 1, 2, \ldots$$

exists in the matrix space defined in Subsection 5.1.

For the proof see Appendix A.

Proposition 5.5. *If inequality (43) holds, then*

$$\lim_{T\to\infty} \prod_{k=1}^{T+1} L_{T,k}^{-1} = 0 \tag{46}$$

and

$$\lim_{T\to\infty} g_{T,T} = g_{\infty}, \tag{47}$$

where g_{∞} is some vector in $\mathbb{R}^{n_y}$.

Proof. From (39) and 5.4 it follows that

$$\lim_{T\to\infty} L_{T,k} = B_k + K_{\infty,k}Q_{12,k} = L_{\infty,k}.$$

Then the limit in (46) can be represented as

$$\lim_{T\to\infty} \prod_{k=1}^{T+1} L_{T,k}^{-1} = \lim_{T\to\infty} \prod_{k=1}^{T+1} L_{\infty,k}^{-1}. \tag{48}$$

Since $K_{\infty,k}$ is bounded (it follows from formula (A.7) in Appendix A) and

$$\lim_{k\to\infty} Q_{12,k} = 0, \quad \text{and} \quad \lim_{k\to\infty} B_k^{-1} = B^{-1},$$

we have $lim_{k\to\infty} L_{\infty,k}^{-1} = B^{-1}$. Therefore, if $\delta > 0$ is arbitrary small, there is an $N = N_\delta \in \mathbb{N}$ such that

$$\|L_{\infty,k}^{-1}\| \leq \beta + \delta = \rho < 1, \tag{49}$$

for $k > N$, where β is the largest eigenvalue (in modulus) of the matrix B^{-1}. From this, the norm property and (48) we obtain

$$\lim_{T\to\infty} \left\| \prod_{k=1}^{T+1} L_{T,k}^{-1} \right\| \leq \lim_{T\to\infty} \prod_{k=1}^{T+1} \left\| L_{\infty,k}^{-1} \right\| \leq \lim_{T\to\infty} C_1 \rho^{T-K} = 0,$$

where C_1 is some constant.

By (49) the products in (41) decay exponentially with the factor ρ as $j \to \infty$. From this and the boundedness of the terms $K_{T,k}$, $\Psi_{2,k}$, $\Psi_{1,k}$ and $E_0\eta_k$, $T \in \mathbb{N}$ and $k = 1, 2, \ldots, T+1$, it follows that the series

$$g_{T,T} = -\sum_{j=1}^{T+1} \prod_{k=1}^{j} L_{T,k}^{-1} (\Psi_{2,j} + K_{T,j}\Psi_{1,j}) E_0 \eta_j.$$

converges to some g_∞ as $T \to \infty$. □

From Proposition 5.4 and Proposition 5.5 it may be concluded that as T tends to infinity Equation (45) takes the form:

$$u_0 = -K_{\infty,0} s_0 + g_\infty. \tag{50}$$

Formula (50) provides a unique bounded solution to the transformed rational expectation model with time-varying parameters (36)–(37), and may be treated as a policy function for this type of problems.

Remark 1. Particularly, for $n = 1$ we have

$$\begin{aligned} g_{T,T}^{(1)} &= -\sum_{j=1}^{T+1} \prod_{k=1}^{j} L_{T,k}^{-1}(\Psi_{2,j} + K_{T,j}\Psi_{1,j})E_0 z_j^{(1)} \\ &= -\sum_{j=1}^{T+1} \prod_{k=1}^{j} L_{T,k}^{-1}(\Psi_{2,j} + K_{T,j}\Psi_{1,j})\Lambda^{j+1} z_0^{(1)}. \end{aligned} \tag{51}$$

Taking into account (8), we get $g_{T,T}^{(1)} = 0$.

Remark 2. The details of derivations for the solution of time-varying rational expectations model corresponding to the first order approximation of the system (15) and (16) are carried out in Appendix B, where we also derive the moving-average representation for $x_t^{(1)}$ and $y_t^{(1)}$. Having this representation it is not hard to compute all quadratic terms in (17).

Remark 3. If $c = 0$ or $d = 0$ (or both) in the inequality (42), i.e. one of the variables s_t or u_t (or both) is exogenous to the other, then (42) is always valid under the conditions (35).

Remark 4. The inequality (42) is a sufficient condition for the existence of the solution in the form (50), and can be weakened. For the representation (50) we need only the invertibility of matrices $L_{T,T-i}$ defined in (39).

5.3. Initial conditions.

It remains to find the initial condition for a stable solution to the system (36)–(37) corresponding the initial condition (16). Recall that we deal with the n-order problem (15)–(16), and we now put the superscript (n) back in notation. From (26) and (50) we have

$$\left[\begin{array}{c} s_0^{(n)} \\ -K_{\infty,0}^{(n)} s_0^{(n)} + g_\infty^{(n)} \end{array} \right] = Z^{-1} \left[\begin{array}{c} 0 \\ y_0^{(n)} \end{array} \right],$$

where Z^{-1} is a matrix that is involved in the block-diagonal factorization (24) and has the following block-decomposition:

$$Z^{-1} = \left[\begin{array}{cc} Z^{11} & Z^{12} \\ Z^{21} & Z^{22} \end{array} \right].$$

Hence

$$s_0^{(n)} = Z^{12} y_0^{(n)}, \tag{52}$$

$$-K_{\infty,0}^{(n)} s_0^{(n)} + g_\infty^{(n)} = Z^{22} y_0^{(n)}. \tag{53}$$

Substituting (52) into (53) , we get

$$y_0^{(n)} = (Z^{22} + K_{\infty,0}^{(n)} Z^{12})^{-1} g_\infty^{(n)}. \tag{54}$$

The vector $(y_0^{(n)}, 0)$ is the initial condition corresponding to a bounded solution to (15) for $t > 0$, hence formula (54) determines the solution to the original rational expectations model with time-varying parameters. In other words, $y_0^{(n)}$ is a policy function for the rational expectations model with time-varying parameters at the point $x_0^{(n)} = 0$. Particularly, for $n = 1$ from (54) and taking into account $g_{T,T}^{(1)} = 0$ we have $y_0^{(1)} = 0$. The condition of the invertibility of matrix $Z^{22} + K_{\infty,0}^{(n)} Z^{12}$ corresponds to Proposition 1 of Blanchard, and Kahn (1980).

5.4. Expected dynamics. Restoring the original variables $x_t^{(n)}$ and $y_t^{(n)}$.

To compute the expected dynamics (impulse response function) it is more convenient to work with auxiliary variables $u_t^{(n)}$ and $s_t^{(n)}$, then to restore the original variables $x_t^{(n)}$ and $y_t^{(n)}$. Substituting (50) for u_t in (36) and taking expectations at $t = 0$ gives

$$E_0 s_{t+1}^{(n)} = (A_t - Q_{12,t} K_{\infty,t}^{(n)}) E_0 s_t^{(n)} + Q_{12,t} g_\infty^{(n)} + \Psi_{1,t} E_0 \eta_{t+1}^{(n)}. \tag{55}$$

From (52) we can compute the initial condition $s_0^{(n)}$ for (55). Knowing the initial value $s_0^{(n)}$ allows us to obtain the whole trajectory of the solution to (55), i.e. $E_0 s_t, t \in \mathbb{N}$. The expected dynamics of $E_0 u_t$ can easily be obtained from (50)

$$E_0 u_t = -K_{\infty,t}^{(n)} E_0 s_t + g_\infty. \tag{56}$$

Then the expected dynamics of the original variables is restored by

$$E_0 x_t^{(n)} = Z_{11} E_0 s_t^{(n)} + Z_{12} E_0 u_t^{(n)}, \tag{57}$$

$$E_0 y_t^{(n)} = Z_{21} E_0 s_t^{(n)} + Z_{22} E_0 u_t^{(n)}, \tag{58}$$

where Z_{ij}, $i = 1, 2$, $j = 1, 2$, are blocks of the block-decomposition of the matrix Z. From (55) and (56) it follows that the process (s_t, u_t) is stable, since $A_t \to A$ as $t \to \infty$, but A is a stable matrix, $Q_{12,t} \to 0$ as $t \to \infty$, and $K_{\infty,t}^{(n)}$ are bounded matrices for $t \geq 0$. From this and (57) and (58) it may be concluded that the process $(y_t^{(n)}, x_t^{(n)})$ is also stable. To sum up, under the assumption that the solutions of lower order than n are already computed in the same manner as for the nth order, we find the stable solution to the original model (1) in the form

$$E_0 y_t = \sum_{i=0}^{n} \sigma^i E_0 y_t^{(i)},$$

$$E_0 x_t = \sum_{i=0}^{n} \sigma^i E_0 x_t^{(i)}.$$

6. An Asset Pricing Model

To illustrate how the presented method works we apply it to an nonlinear asset pricing model proposed by Burnside (1998) and analyzed by Collard, and Juillard (2001); Schmitt-Grohé, and Uribe (2004). The simplicity of the model allows us to derive all approximations in the analytical form.

The representative agent maximizes the lifetime utility function

$$\max\left(E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^\theta}{\theta}\right)$$

subject to

$$p_t e_{t+1} + C_t = p_t e_t + d_t e_t,$$

where $\beta > 0$ is a subjective discount factor, $\theta < 1$ and $\theta \neq 0$, C_t denotes consumption, p_t is the price at date t of a unit of the asset, e_t represents units of a single asset held at the beginning of period t, and d_t is dividends per asset in period t. The growth of rate of the dividends follows an AR(1) process[7]

$$\mathrm{x_t} = (1-\rho)\,\bar{\mathrm{x}} + \rho x_{t-1} + \sigma \varepsilon_{t+1}, \tag{59}$$

where $x_t = ln(d_t/d_{t-1})$, and $\varepsilon_{t+1} \sim NIID(0,1)$. The first order condition and market clearing yields the equilibrium condition

$$y_t = \beta E_t \left[\exp(\theta x_{t+1})\,(1 + y_{t+1})\right], \tag{60}$$

[7]By abuse of our previous notation, we let x_t stand for the exogenous process as in Burnside (1998); Collard, and Juillard (2001); Schmitt-Grohé, and Uribe (2004).

where $y_t = p_t/d_t$ is the price-dividend ratio. This equation has an exact solution of the form (Burnside (1998))

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp\left[a_i + b_i(x_t - \bar{x})\right], \tag{61}$$

where

$$a_i = \theta\bar{x}i + \frac{1}{2}\left(\frac{\theta\sigma}{1-\rho}\right)^2 \left[i - \frac{2\rho(1-\rho^i)}{1-\rho} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right] \tag{62}$$

and

$$b_i = \frac{\theta\rho(1-\rho^i)}{1-\rho}.$$

It follows from (60) that the deterministic steady state of the economy is

$$\bar{y} = \frac{\beta\exp(\theta\bar{x})}{1-\beta\exp(\theta\bar{x})}.$$

6.1. Approximate solution

We now obtain a solution to the system (59)–(60) as an expansion in powers of the parameter σ using the second-order approximation method developed in Sections 3–5. Specifically, we are seeking for the solution of the form:

$$y_t = y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)} \tag{63}$$

$$x_t = x_t^{(0)} + \sigma x_t^{(1)}. \tag{64}$$

Substituting (64) into (59) and collecting the terms containing σ^0 and σ^1, we obtain the representation (64) for x_t

$$x_{t+1}^{(0)} = (1-\rho)\,\bar{\mathrm{x}}\, + \rho x_t^{(0)} \tag{65}$$

$$x_{t+1}^{(1)} = \rho x_t^{(1)} + \varepsilon_{t+1}. \tag{66}$$

Since the expansion (64) must be valid for all σ at the initial time $t = 0$, the initial conditions are

$$x_0^{(0)} = x_0 \quad \text{and} \quad x_0^{(1)} = 0. \tag{67}$$

Substituting (63) and (64) into (60), expanding in series for small σ, then collecting the terms of like powers of σ and setting the coefficients of like powers of σ to zero, we have (for details see Appendix C)

Coefficient of σ^0

$$y_t^{(0)} = \beta \exp(\theta x_{t+1}^{(0)})(1 + y_{t+1}^{(0)}), \tag{68}$$

$$x_{t+1}^{(0)} = \rho x_t^{(0)}. \tag{69}$$

Coefficient of σ^1

$$y_t^{(1)} = \exp(\theta x_{t+1}^{(0)})\beta E_t \left[\theta x_{t+1}^{(1)}(1 + y_{t+1}^{(0)}) + y_{t+1}^{(1)}\right], \tag{70}$$

$$x_{t+1}^{(1)} = \rho x_t^{(1)} + \varepsilon_{t+1}. \tag{71}$$

Coefficient of σ^2

$$\begin{aligned} y_t^{(2)} &= \tfrac{1}{2}\beta \exp(\theta x_{t+1}^{(0)})\theta^2 \left(1 + y_{t+1}^{(0)}\right) E_t \left(x_{t+1}^{(1)}\right)^2 \\ &+\beta \exp(\theta x_{t+1}^{(0)}) E_t \left(\theta x_{t+1}^{(1)} y_{t+1}^{(1)}\right) + \beta \exp(\theta x_{t+1}^{(0)}) E_t \left(y_{t+1}^{(2)}\right). \end{aligned} \tag{72}$$

The system (68) and (69) is a deterministic model. Its solution can easily be obtained by, for example, forward induction

$$y_t^{(0)} = \sum_{i=1}^{\infty} \beta^i \exp\left\{\theta \left[\bar{x} i + \frac{\rho(1-\rho^i)}{1-\rho}(x_t - \bar{x})\right]\right\}. \tag{73}$$

Under the assumption that $y_t^{(0)}$ and $x_t^{(0)}$ are known for $t \geq 0$, Equations (70) and (71) constitute a linear rational expectations model with time varying deterministic coefficients that can be solved by the backward recursion method considered in Section 5. Assuming that the terminal moment $T = \infty$, we have

$$y_t^{(1)} = \underbrace{\theta \left[\sum_{i=1}^{\infty} (\beta\rho)^i \left(1 + y_{t+i}^{(0)}\right) exp \left(\theta \sum_{j=1}^{i} x_{t+j}^{(0)}\right)\right]}_{=-K_{\infty,t}} x_t^{(1)} = -K_{\infty,t} x_t^{(1)}. \tag{74}$$

Note that $x_0^{(1)} = 0$, hence $y_0^{(1)} = 0$.

Equation (72) is also a linear forward-looking equation with time varying deterministic coefficients. Substituting (74) for $y_t^{(1)}$ in (72) and collecting the terms with $E_t \left(x_{t+1}^{(1)}\right)^2$ yields

$$\begin{aligned} y_t^{(2)} &= \tfrac{1}{2}\beta \exp(\theta x_{t+1}^{(0)})\theta \left[\theta \left(1 + y_{t+1}^{(0)}\right) - 2K_{\infty,t+1}\right] E_t \left(x_{t+1}^{(1)}\right)^2 \\ &+ \exp(\theta x_{t+1}^{(0)}) E_t \left(y_{t+1}^{(2)}\right), \end{aligned} \tag{75}$$

This equation can also be solved by the backward recursion considered in Section 5. It is easily shown that the solution of (75) has the form

$$y_t^{(2)} = \frac{1}{2}\theta \sum_{i=1}^{\infty} \beta^i \exp\left(\theta \sum_{j=1}^{i} x_{t+j}^{(0)}\right) \left[\theta\big(1 + y_{t+i}^{(0)}\big) - 2K_{\infty,t+i}\right] E_t\big(x_{t+i}^{(1)}\big)^2 \quad (76)$$

From (71) and (67) we have the moving-average representation for $x_{t+1}^{(1)}$:

$$x_{t+i}^{(1)} = \varepsilon_{t+i} + \rho\varepsilon_{t+i-1} + ... + \rho^{i-1}\varepsilon_{t+1}.$$

Since the sequence of innovations ε_t, $t > 0$, is independent it follows that

$$\begin{aligned} E_t\left(x_{t+i}^{(1)}\right)^2 &= E_t\left(\varepsilon_{t+i} + \rho\varepsilon_{t+i-1} + \cdots + \rho^{i-1}\varepsilon_{t+1}\right)^2 \\ &= 1 + \rho^2 + \cdots + \rho^{2(i-1)} = \frac{1-\rho^{2i}}{1-\rho^2}. \end{aligned} \quad (77)$$

From (65) we have

$$\begin{aligned} &x_{t+1}^{(0)} + x_{t+2}^{(0)} + \cdots + x_{t+i}^{(0)} = \bar{x} + \rho(x_t^{(0)} - \bar{x}) + \bar{x} + \rho^2(x_t^{(0)} - \bar{x}) \\ &+\bar{x} + \rho^i(x_t^{(0)} - \bar{x}) = i\bar{x} + \tfrac{\rho(1-\rho^i)}{1-\rho}(x_t^{(0)} - \bar{x}). \end{aligned} \quad (78)$$

Finally, inserting (77) and (78) into (76) gives

$$y_t^{(2)} = \frac{\theta}{2}\sum_{i=1}^{\infty} \beta^i \frac{1-\rho^{2i}}{1-\rho^2} \exp\left\{\theta\left[i\bar{x} + b(x_t^{(0)} - \bar{x})\right]\right\} \left[\theta(1 + y_{t+i}^{(0)}) - 2K_{\infty,t+1}\right].$$

From Remarks 3 and 4 it follows that the solution in this form exists for any initial condition x_0. To summarize, we find the policy function approximation in the form

$$y(x_0) = h(x_0) = y_t^{(0)}(x_0) + \sigma^2 y_t^{(2)}(x_0).$$

The solutions for the higher orders $y_t^{(n)}(x)$, $n > 2$, can be obtained in much the same way as for $y_t^{(2)}(x)$.

6.2. Comparison with the local perturbation

This subsection compares the policy functions of the second order of the presented method with the local Taylor series expansions of order two (Schmitt-Grohé, and Uribe (2004)).

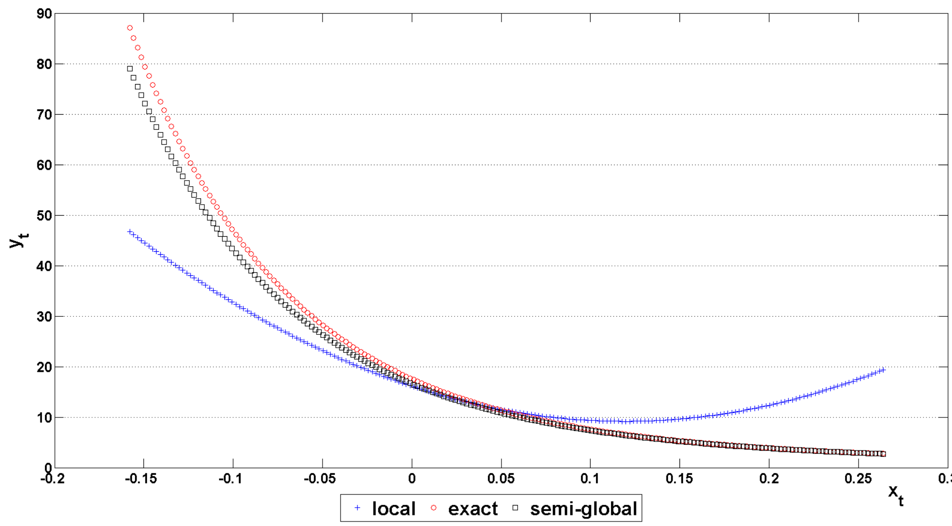

Figure 1: Comparison of the policy functions.

The parameterization follows Collard, and Juillard (2001), where the benchmark parameterization is chosen as in Mehra, and Prescott (1985). We therefore set the mean of the rate of growth of dividend to $\bar{x} = 0.0179$, the volatility of the innovations to $\sigma = 0.015$, the parameter θ to -1.5 and β to 0.95. For illustrative purpose, we choose the highly persistent exogenous process with $\rho = 0.9$.

Figure 1 illustrates the exact policy function with the approximate ones constructed by the semi-global method and the local Taylor series expansions. The figure shows that the semi-global method traces globally the pattern of the true policy function much better than the local Taylor series expansion. Moreover, from Figure 1 we can also see another undesirable property of the the local Taylor series expansion, namely this method may provide impulse response functions with wrong signs. Indeed, the steady state value of y_t is $\bar{y} = 12.3$. After a big positive shock the true impulse response function is negative (the policy function values are below the steady state), whereas the impulse response function implied by the local perturbation method is positive (the approximate policy function is above the steady state). Note also that the solution produced by the semi-global method is indistinguishable from the true solution for positive shocks (the bottom right corner of the Figure 1).

7. Conclusion

This study proposes an approach based on a perturbation around a deterministic path for constructing global approximate solutions to DSGE models. Under the assumption that the deterministic solution to the model is already found, the approach reduces the problem to solving recursively a set of linear rational expectations models with deterministic time-varying parameters and the same homogeneous part. The paper also proposes a method to solve linear rational expectations models with deterministic time-varying parameters. The method may be valuable in itself, for example for solving models with anticipated structural changes in parameters. The conditions under which the solutions exist are found; all results are obtained for DSGE models in general form and proved rigorously.

The proposed approach has a potential to solve high-dimensional models as it shares some preferable properties with the local perturbation methods. Namely, the computational gain may come from the calculation of conditional expectations. To compute conditional expectations using the semi-global method all we need is to know the moments of distribution up to the order of approximation, while the use of the global methods, such as projection and simulation, involves either quadrature or stochastic simulations. The former can deal with only low-order integrals, the latter is time consuming. Actually, under the conditions of "smallness" of a scaling parameter and existence of higher order moments for stochastic terms, all derivations of Section 3 hold irrespective of probability distribution functions for these stochastic terms.

The paper illustrates the algorithm up to order two using an nonlinear asset pricing model by Burnside (1998) and compares it with the local Taylor series expansion. The author leaves the practical implementation of the approach to models of larger size for future research.

Appendix A. Proofs for Section 5

PROOF OF PROPOSITION 5.1: The proof is by induction on i. Suppose that $i = 0$. For the time T from (37) we have

$$E_T u_{T+1} = B_T u_T + Q_{21,T} s_T + \Psi_{2,T} E_T \eta_{T+1}.$$

As B_T is invertible, we have

$$u_{T,T} = -K_{T,T} s_T - g_{T,0} + L_{T,T}^{-1} E_T u_{T+1},$$

where $K_{T,T} = B_T^{-1} Q_{21,T}$; $g_{T,0} = -B_T^{-1}\Psi_{2,T} E_T \eta_{T+1}$ and $L_{T,T}^{-1} = B_T^{-1}$. From (38), (39) and (41) it follows that the inductive assumption is proved for $i = 0$. Assuming that (40) holds for $i > 0$, we will prove it for $i + 1$. To this end, consider Equation (37) for the time $t = T - i - 1$. As the matrix B_{T-i} is invertible, we obtain

$$\begin{aligned} u_{T,T-i-1} = -B_{T-i-1}^{-1} Q_{21,T-i-1} s_{T-i-1} - B_{T-i-1}^{-1} \Psi_{2,T-i-1} E_{T-i-1} \eta_{T-i} \\ + B_{T-i-1}^{-1} E_{T-i-1} u_{T,T-i}. \end{aligned}$$

Substituting the induction assumption (40) for $u_{T,T-i}$ yields

$$\begin{aligned} &u_{T,T-i-1} = -B_{T-i-1}^{-1} Q_{21,T-i-1} s_{T-i-1} - B_{T-i-1}^{-1} \Psi_{2,T-i-1} E_{T-i-1} \eta_{T-i} \\ &+ B_{T-i-1}^{-1} E_{T-i-1} \left[-K_{T,T-i} s_{T-i} + g_{T,i} + \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1} \right) E_{T-i} \left(u_{T+1} \right) \right]. \end{aligned}$$

Substituting (36) for $E_{T-i-1}(s_{T-i})$ and using the law of iterated expectations gives

$$\begin{aligned} &u_{T,T-i-1} = -B_{T-i}^{-1} Q_{21,T-i} s_{T-i-1} - B_{T-i}^{-1} \Psi_{2,T-i} E_{T-i-1} \eta_{T-i} + B_{T-i}^{-1} g_{T,i} \\ &+ B_{T-i}^{-1} \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1} \right) E_{T-i-1} \left(u_{T+1} \right) \\ &+ B_{T-i}^{-1} \left[-K_{T,T-i} \left(A_{T-i} s_{T-i-1} + Q_{12,T-i} u_{T,T-i-1} + \Psi_{1,T-i} E_{T-i-1} \eta_{T-i} \right) \right]. \end{aligned}$$

Collecting the terms with $u_{T,T-i-1}$, s_{T-i-1} and η_{T-i}, we get

$$\begin{aligned} &\left(I + B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i} \right) u_{T,T-i-1} = -B_{T-i}^{-1} \big[(Q_{21,T-i} + K_{T,T-i} A_{T-i}) s_{T-i-1} \\ &+ (\Psi_{2,T-i} + K_{T,T-i} \Psi_{1,T-i}) E_{T-i-1} \eta_{T-i} + g_{T,i} + \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1} \right) E_{T-i-1} \left(u_{T+1} \right) \big] \end{aligned}$$

Suppose for the moment that the matrix $Z_{T,T-i} = I + B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i}$ is invertible. Pre-multiplying the last equation by $Z_{T,T-i}^{-1}$, we obtain

$$\begin{aligned} &u_{T,T-i-1} = -Z_{T,T-i}^{-1} B_{T-i}^{-1} \big[(Q_{21,T-i} + K_{T,T-i} A_{T-i}) s_{T-i-1} \\ &+ (\Psi_{2,T-i} + K_{T,T-i} \Psi_{1,T-i}) E_{T-i-1} \eta_{T-i} + g_{T,i} \\ &+ \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1} \right) E_{T-i-1} \left(u_{T+1} \right) \big]. \end{aligned}$$

Note that $L_{T,T-i} = B_{T-i} Z_{T,T-i}$; then using the definition of $K_{T,T-i-1}$ (38), we see that

$$\begin{aligned} &u_{T,T-i-1} = -K_{T,T-i-1} s_{T-i-1} \\ &- L_{T,T-i}^{-1} \left(\Psi_{2,T-i} + K_{T,T-i} \Psi_{1,T-i} \right) E_{T-i-1} \eta_{T-i} \\ &+ L_{T,T-i}^{-1} g_{T,i} + L_{T,T-i}^{-1} \left(\prod_{k=1}^{i+1} L_{T,T-i+k}^{-1} \right) E_{T-i-1} \left(u_{T+1} \right). \end{aligned} \tag{A.1}$$

Using the definition of $g_{T,i}$ and $L_{T-i,T-i+j}$ ((39) and (41)), we deduce that

$$g_{T,i+1} = -L_{T,T-i}^{-1} \left(\Psi_{2,T-i} + K_{T,T-i} \Psi_{1,T-i} \right) E_{T-i-1} \eta_{T-i} + L_{T,T-i}^{-1} g_{T,i}. \tag{A.2}$$

From (A.1) and (A.2) it follows that

$$u_{T,T-i-1} = -K_{T,T-i-1} s_{T-i-1} + g_{T,i+1} + \left(\prod_{k=1}^{i+2} L_{T,T-i-1+k}^{-1} \right) E_{T-i-1} \left(u_{T+1} \right).$$

□

PROOF OF PROPOSITION 5.2: We begin by rewriting (38) as

$$\left(B_{T-i} + K_{T,T-i} Q_{12,T-i} \right) K_{T,T-(i+1)} = \left(Q_{21,T-i} + K_{T,T-i} A_{T-i} \right).$$

Rearranging terms, we have

$$\begin{aligned} K_{T,T-(i+1)} = {} & B_{T-i}^{-1} \cdot \left(Q_{21,T-i} + K_{T,T-i} A_{T-i} \right) \\ & - B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i} K_{T,T-(i+1)}. \end{aligned} \tag{A.3}$$

Taking the norms and using the norm properties gives

$$\begin{aligned} \left\| K_{T,T-(i+1)} \right\| \leq {} & \left\| B_{T-i}^{-1} \right\| \cdot \left\| Q_{21,T-i} \right\| + \left\| B_{T-i}^{-1} \right\| \cdot \left\| K_{T,T-i} \right\| \cdot \left\| A_{T-i} \right\| \\ & + \left\| B_{T-i}^{-1} \right\| \cdot \left\| K_{T,T-i} \right\| \cdot \left\| Q_{12,T-i} \right\| \cdot \left\| K_{T,T-(i+1)} \right\|. \end{aligned}$$

Rearranging terms, we get

$$\left\| K_{T,T-(i+1)} \right\| \leq \frac{\left\| B_{T-i}^{-1} \right\| \cdot \left\| Q_{21,T-i} \right\| + \left\| B_{T-i}^{-1} \right\| \cdot \left\| K_{T,T-i} \right\| \cdot \left\| A_{T-i} \right\|}{1 - \left\| B_{T-i}^{-1} \right\| \cdot \left\| K_{T,T-i} \right\| \cdot \left\| Q_{12,T-i} \right\|}. \tag{A.4}$$

Inequality (A.4) is a difference inequality with respect to $\left\| K_{T,T-i} \right\|$, $i = 0, 1, \ldots, T$, and with the time-varying coefficients $\left\| A_{T-i} \right\|$, $\left\| B_{T-i}^{-1} \right\|$, $\left\| Q_{12,T-i} \right\|$ and $\left\| Q_{21,T-i} \right\|$. In (A.4) we assume that
$1 - \left\| B_{T-i}^{-1} \right\| \cdot \left\| K_{T,T-i} \right\| \cdot \left\| Q_{12,T-i} \right\| \neq 0$.

This is obviously true if $\left\| K_{T,T-i} \right\| = 0$. We shall show that if the initial condition $\left\| K_{T,T+1} \right\| = 0$, then $\left(1 - \left\| B_{T-i}^{-1} \right\| \cdot \left\| K_{T,T-i} \right\| \cdot \left\| Q_{12,T-i} \right\| \right) > 0$, $i = 1, 2, \ldots, T$. Indeed, consider the difference equation:

$$s_{i+1} = \frac{bd + bas_i}{\left(1 - bcs_i \right)}. \tag{A.5}$$

Lemma Appendix A.1. *If inequality (42) holds, then the difference equation (A.5) has two fixed points*

$$s_1^* = \frac{2bd}{1 - ba + \sqrt{(1-ba)^2 - 4b^2cd}}, \tag{A.6}$$

$$s_2^* = \frac{1 - ba + \sqrt{(1-ba)^2 - 4b^2cd}}{2bc},$$

where s_1^* is a stable fixed point whereas s_2^* is an unstable one. Moreover, under the initial condition $s_0 = 0$ the solution $s_i, i = 1, 2, \ldots,$ is an increasing sequence and converges to s_1^*.

The lemma can be proved by direct calculation. From (33)–(32) the values a, b, c and d majorize $\|A_{T-i}\|$, $\left\|B_{T-i}^{-1}\right\|$, $\|Q_{12,T-i}\|$ and $\|Q_{21,T-i}\|$, respectively. If we consider Equation (A.2) and inequality (A.5) as initial value problems with the initial conditions $\|K_{T,T+1}\| = 0$ and $s_0 = 0$, then their solutions obviously satisfy the inequality $\|K_{T,T-i}\| \leq s_{i+1}$, $i = 1, 2, \ldots, T$. In other words, $\|K_{T,T-i}\|$ is majorized by s_i. From the last inequality and Lemma Appendix A.1 it may be concluded that

$$\|K_{T,T-i}\| \leq s_1^*, \quad i = 0, 1, 2, \ldots, T, \quad T \in \mathbb{N}. \tag{A.7}$$

From (A.6), (A.7) and (33) it follows that

$$\left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\| \leq \frac{2b^2dc}{1 - ba + \sqrt{(1-ba)^2 - 4b^2cd}}. \tag{A.8}$$

From (42) we see that $2b^2dc < (1-ab)^2/2$. Substituting this inequality into (A.8) gives

$$\begin{aligned} \left\|B_{T-i}^{-1}\right\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\| &\leq \frac{(1-ba)^2}{2(1 - ba + \sqrt{(1-ba)^2 - 4b^2cd})} \\ &< \frac{(1-ba)^2}{2(1-ba)} = \frac{1-ba}{2} < 1, \end{aligned} \tag{A.9}$$

where the last inequality follows from (35). □

PROOF OF PROPOSITION 5.4: The assertion of the proposition is true if there exist constants M and r such that $0 < r < 1$ and for $T \in \mathbb{N}$

$$\|K_{T,j} - K_{T+1,j}\| \leq Mr^{T+1}, \quad j = 0, 1, 2, \ldots. \tag{A.10}$$

Note now that $K_{T,j}$ ($K_{T+1,j}$) is a solution to the matrix difference equation (38) at $i = T - j$ ($i = T + 1 - j$) with the initial condition $K_{T,T+1} = 0$ ($K_{T+1,T+2} = 0$). Subtracting (A.3) for $K_{T,T-(i+1)}$ from that for $K_{T+1,T-(i+1)}$, we have

$$K_{T,T-(i+1)} - K_{T+1,T-(i+1)} = B_{T-i}^{-1}(K_{T,T-i)} - K_{T+1,T-i})A_{T-i}$$
$$- B_{T-i}^{-1}K_{T,T-i)}Q_{12,T-i}K_{T,T-(i+1)} + B_{T-i}^{-1}K_{T+1,T-i}Q_{12,T-i}K_{T+1,T-(i+1)}.$$

Adding and subtracting $B_{T-i}^{-1} \cdot K_{T,T-i} \cdot Q_{12,T-i} \cdot K_{T+1,T-(i+1)}$ in the right hand side gives

$$K_{T,T-(i+1)} - K_{T+1,T-(i+1)} = B_{T-i}^{-1}(K_{T,T-i)} - K_{T+1,T-i})A_{T-i}$$
$$- B_{T-i}^{-1} \cdot K_{T,T-i} \cdot Q_{12,T-i}(K_{T,T-(i+1)} - K_{T+1,T-(i+1)})$$
$$- B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})Q_{12,T-i} \cdot K_{T+1,T-(i+1)}.$$

Rearranging terms yields

$$(I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i})(K_{T,T-(i+1)} - K_{T+1,T-(i+1)})$$
$$= B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})A_{T-i}$$
$$- B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})Q_{12,T-i}K_{T+1,T-(i+1)}.$$

From Proposition 5.3 it follows that the matrix

$$Z_{T,T-i} = (I + B_{T-i}^{-1}K_{T,T-i}Q_{12,T-i})$$

is invertible, then pre-multiplying the last equation by this matrix yields

$$K_{T,T-(i+1)} - K_{T+1,T-(i+1)} = Z_{T,T-i}^{-1}(B_{T-i}^{-1}(K_{T,T-i} - K_{T+1,T-i})A_{T-i}$$
$$- B_{T-i}^{-1}(K_{T,T-i)} - K_{T+1,T-i})Q_{12,T-i}K_{T+1,T-(i+1)}).$$

Taking the norms, using the norm property and the triangle inequality, we get

$$\begin{aligned} &\|K_{T,T-(i+1)} - K_{T+1,T-(i+1)}\| \\ &\leq \|Z_{T,T-i}^{-1}\| \cdot (\|B_{T-i}^{-1}\| \cdot \|K_{T,T-i} - K_{T+1,T-i}\| \cdot \|A_{T-i}\| \\ &+ \|B_{T-i}^{-1}\| \cdot \|K_{T,T-i)} - K_{T+1,T-i}\| \cdot \|Q_{12,T-i}\| \cdot \|K_{T+1,T-(i+1)}\|). \end{aligned} \tag{A.11}$$

From (32) and (A.9) we have

$$\begin{aligned} &\|K_{T,T-(i+1)} - K_{T+1,T-(i+1)}\| \\ &\leq \left(ab + \frac{1-ba}{2}\right)\|Z_{T,T-i}^{-1}\| \cdot \|K_{T,T-i} - K_{T+1,T-i}\| \\ &= \frac{1+ba}{2}\|Z_{T,T-i}^{-1}\| \cdot \|K_{T,T-i} - K_{T+1,T-i}\|. \end{aligned} \tag{A.12}$$

From the norm property and Golub, and Van Loan (1996, Lemma 2.3.3) we get the estimate

$$\|Z_{T,T-i}^{-1}\| = \|(I + B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i})^{-1}\| \leq \frac{1}{1 - \|B_{T-i}^{-1} K_{T,T-i} Q_{12,T-i}\|}$$
$$\leq \frac{1}{1 - \|B_{T-i}^{-1}\| \cdot \|K_{T,T-i}\| \cdot \|Q_{12,T-i}\|}$$

By (A.9), we have

$$\|Z_{T,T-i}^{-1}\| =< \frac{1}{1 - \frac{1-ba}{2}} = \frac{2}{1+ba}$$

Substituting the last inequality into (A.12) gives

$$\|K_{T,T-(i+1)} - K_{T+1,T-(i+1)}\| < \|K_{T,T-i} - K_{T+1,T-i}\|. \tag{A.13}$$

Using (A.16) successively for $i = -1, 0, 1, \ldots, T-1$, and taking into account $K_{T,T+1} = 0$ and $K_{T+1,T+1} = B_{T+2}^{-1} Q_{21,T+2}$ results in

$$\begin{aligned} &\|K_{T,j} - K_{T+1,j}\| < \|K_{T,T+1} - K_{T+1,T+1}\| = \|B_{T+2}^{-1} Q_{21,T+2}\| \\ &\leq \|B_{T+2}^{-1}\| \cdot \|Q_{21,T+2}\| \leq b\|Q_{21,T+2}\|, \quad j = 0, 1, 2, \ldots. \end{aligned} \tag{A.14}$$

Recall that $Q_{21,T}$ depends on the solution to the deterministic problem (10), i.e.

$$Q_{21,T} = Q_{21}\left(x_{T+1}^{(0)}, x_T^{(0)}, z_{T+1}^{(0)}, z_T^{(0)}\right).$$

From Hartmann (1982, Corollary 5.1) and differentiability of Q_{21} with respect to the state variables it follows that

$$\|Q_{21,T}\| \leq C(\alpha + \theta)^T, \tag{A.15}$$

where α is the largest eigenvalue modulus of the matrix A from (25), C is some constant and θ is arbitrary small positive number. In fact, $\alpha + \theta$ determines the speed of convergence for the deterministic solution to the steady state. Inserting (A.15) into (A.16), we can conclude

$$\|K_{T,j} - K_{T+1,j}\| < bC(\alpha + \theta)^{T+2}, \quad j = 0, 1, 2, \ldots \tag{A.16}$$

Denoting $M = bC(\alpha + \theta)$ and $r = \alpha + \theta$ we finally obtain (A.10). □

Appendix B. The First Order System

For $n = 1$ we have

$$\begin{bmatrix} s_0^{(1)} \\ u_0^{(1)} \end{bmatrix} = Z_1 Z^{-1} \begin{bmatrix} x_0^{(1)} \\ y_0^{(1)} \end{bmatrix} = 0,$$

From (28) for the time T we have

$$u_T^{(1)} = -B_{T+1}^{-1} Q_{21,T+1} s_T^{(1)} - B_{T+1}^{-1} \Pi_{2,t+1} z_T^{(1)} + B_{T+1}^{-1} E_T u_{T+1}^{(1)}.$$

Denoting $K_{T,T} = B_{T+1}^{-1} Q_{21,T+1}$ and $R_T = B_{T+1}^{-1} \Pi_{2,t+1}$ gives

$$u_T^{(1)} = -K_{T,T} s_T^{(1)} - R_T z_T^{(1)} + B_{T+1}^{-1} E_T u_{T+1}^{(1)}. \tag{B.1}$$

For $T-1$ we have

$$u_{T-1}^{(1)} = -B_T^{-1} Q_{21,T} s_{T-1}^{(1)} - B_T^{-1} \Pi_{2,t} z_{T-1}^{(1)} + B_T^{-1} E_{T-1} u_T^{(1)}. \tag{B.2}$$

Taking conditional expectations at the time $T-1$ from both side (B.1) and inserting (2) we get

$$E_{T-1} u_T^{(1)} = -K_{T,T} E_{T-1} s_T^{(1)} - R_T \Lambda z_{T-1}^{(1)} + B_{T+1}^{-1} E_{T-1} u_{T+1}^{(1)}. \tag{B.3}$$

Inserting (B.3) into (B.2) gives

$$\begin{aligned} u_{T-1}^{(1)} &= -B_T^{-1} Q_{21,T} s_{T-1}^{(1)} - B_T^{-1} \Pi_{2,t} z_{T-1}^{(1)} \\ &+ B_T^{-1} E_{T-1}(-K_{T,T} E_{T-1} s_T - R_T \Lambda z_{T-1}^{(1)} + B_{T+1}^{-1} E_T u_{T+1}^{(1)}). \end{aligned} \tag{B.4}$$

Inserting now $E_{T-1} s_T$ into (B.4) from (27) yields

$$\begin{aligned} u_{T-1}^{(1)} &= -B_T^{-1} Q_{21,T} s_{T-1}^{(1)} - B_T^{-1} \Pi_{2,t} z_{T-1}^{(1)} \\ &+ B_T^{-1} E_{T-1}[-K_{T,T}(A_{T-1} s_{T-1}^{(1)} + Q_{12,T} u_{T-1}^{(1)} + \Pi_{1,T} z_{T-1}^{(1)}) \\ &- R_T \Lambda z_{T-1}^{(1)} + B_{T+1}^{-1} E_T u_{T+1}^{(1)}]. \end{aligned}$$

Reshuffling terms, we have

$$\begin{aligned} (I + B_T^{-1} K_{T,T} Q_{12,T}) u_{T-1}^{(1)} &= -B_T^{-1}(Q_{21,T} + B_T^{-1} K_{T,T} A_{T-1}) s_{T-1}^{(1)} \\ - B_T^{-1}(\Pi_{2,t} + K_{T,T} \Pi_{1,T} + R_T \Lambda) z_{T-1}^{(1)} &+ B_T^{-1} B_{T+1}^{-1} E_T u_{T+1}^{(1)}. \end{aligned} \tag{B.5}$$

Multiplying (B.5) by $(I + B_T^{-1}K_{T,T}Q_{12,T})^{-1}$ yields

$$\begin{aligned}
u_{T-1}^{(1)} &= -(I + B_T^{-1}K_{T,T}Q_{12,T})^{-1}B_T^{-1}(Q_{21,T} + B_T^{-1}K_{T,T}A_{T-1})s_{T-1}^{(1)} \\
&- (I + B_T^{-1}K_{T,T}Q_{12,T})^{-1}B_T^{-1}(\Pi_{2,t} + K_{T,T}\Pi_{1,T} + R_T\Lambda)z_{T-1}^{(1)} \\
&+ (I + B_T^{-1}K_{T,T}Q_{12,T})^{-1}B_T^{-1}B_{T+1}^{-1}E_T u_{T+1}^{(1)}.
\end{aligned}$$

or

$$\begin{aligned}
u_{T-1}^{(1)} &= -(B_T + K_{T,T}Q_{12,T})^{-1}(Q_{21,T} + B_T^{-1}K_{T,T}A_{T-1})s_{T-1}^{(1)} \\
&- (B_T + K_{T,T}Q_{12,T})^{-1}(\Pi_{2,t} + K_{T,T}\Pi_{1,T} + R_T\Lambda)z_{T-1}^{(1)} \\
&+ (B_T + K_{T,T}Q_{12,T})^{-1}B_{T+1}^{-1}E_T u_{T+1}^{(1)}.
\end{aligned}$$

Denoting $L_{T,T-1} = (B_{T-1} + K_{T,T}Q_{12,T-1})$, we obtain

$$\begin{aligned}
u_{T-1}^{(1)} &= -L_{T,T-1}^{-1}(Q_{21,T} + B_T^{-1}K_{T,T}A_{T-1})s_{T-1}^{(1)} \\
&- L_{T,T-1}^{-1}(\Pi_{2,t} + K_{T,T}\Pi_{1,T} + R_T\Lambda)z_{T-1}^{(1)} \\
&+ L_{T,T-1}^{-1}B_{T+1}^{-1}E_T u_{T+1}^{(1)}.
\end{aligned}$$

Denoting

$$K_{T,T-1} = L_{T,T-1}^{-1}(Q_{21,T} + B_T^{-1}K_{T,T}A_{T-1})$$

and

$$R_{T-1} = L_{T,T-1}^{-1}(\Pi_{2,t} + K_{T,T}\Pi_{1,T} + R_T\Lambda),$$

we have

$$u_{T-1}^{(1)} = -K_{T,T-1}s_{T-1}^{(1)} - R_{T-1}z_{T-1}^{(1)} + L_{T,T-1}^{-1}L_{T,T}^{-1}E_T u_{T+1}^{(1)}.$$

Following the same derivation as in Appendix A for the proof of Proposition 5.1, we obtain the following representation:

$$u_t^{(1)} = -K_{T,t}s_t^{(1)} - R_t z_t^{(1)}, \tag{B.6}$$

where R_t can be computed by backward recursion

$$R_t = L_{T,t+1}^{-1}(\Pi_{2,t} + K_{T,t}\Pi_{1,t} + R_{t+1}\Lambda)$$

Inserting (27) into (B.6) gives

$$E_t s_{t+1}^{(1)} = As_t^{(1)} + Q_{11,t+1}s_t^{(1)} + Q_{12,t+1}(-K_{T,t}s_t^{(1)} - R_t z_t^{(1)}) + \Pi_{1,t+1}z_t^{(1)}$$

After reshuffling we get

$$E_t s_{t+1}^{(1)} = (A_{t+1} - Q_{12,t+1} K_{T,t}) s_t^{(1)} + (-Q_{12,t+1} R_t + \Pi_{1,t+1}) z_t^{(1)}.$$

Denoting $\mathbb{A}_t = A_{t+1} - Q_{12,t+1} K_{T,t}$ and $\mathbb{P}_t = -Q_{12,t+1} R_t + \Pi_{1,t+1}$, we have

$$E_t s_{t+1}^{(1)} = \mathbb{A}_t s_t^{(1)} + \mathbb{P}_t z_t^{(1)} \tag{B.7}$$

It is easy to see that

$$\begin{bmatrix} s_{t+1}^{(1)} \\ u_{t+1}^{(1)} \end{bmatrix} - E_t \begin{bmatrix} s_{t+1}^{(1)} \\ u_{t+1}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbb{R}_{1,t} \\ \mathbb{R}_{2,t} \end{bmatrix} \varepsilon_{t+1} = Z \Phi_{t+1}^{-1} f_{5,t+1} \varepsilon_{t+1}.$$

From (B.7) it follows that

$$(E_t s_{t+1} - s_{t+1}) + s_{t+1} = \mathbb{A}_t s_t + \mathbb{P}_t z_t,$$

thus, we obtain

$$s_{t+1} = \mathbb{A}_t s_t + \mathbb{P}_t z_t + \mathbb{R}_{1,t} \varepsilon_{t+1}. \tag{B.8}$$

Recall now that the initial conditions are $s_0^{(1)} = 0$ and $z_0^{(1)} = 0$, then for $t = 1$ from (B.8) we have

$$s_1^{(1)} = \mathbb{R}_{1,0} \varepsilon_1;$$

for $t = 2$

$$s_2^{(1)} = (\mathbb{A}_1 \mathbb{R}_{1,0} + \mathbb{P}_t) \varepsilon_1 + \mathbb{R}_{1,1} \varepsilon_2.$$

Continuing in this fashion, we get the moving-average representation of $s_t^{(1)}$:

$$s_t^{(1)} = \gamma_{t,t} \varepsilon_t + \gamma_{t,t-1} \varepsilon_{t-1} + \cdots + \gamma_{t,2} \varepsilon_2 + \gamma_{t,1} \varepsilon_1, \tag{B.9}$$

where the coefficients $\gamma_{t,t-i}$ can be obtained by forward recursion in $t = 1, 2, \ldots, T$ and backward recursion in $i = 0, 1, \ldots, t-1$

$$\begin{aligned} \gamma_{t,t} &= \mathbb{R}_{1,t-1}, \\ \gamma_{t,t-1} &= \mathbb{A}_{t-1} \gamma_{t-1,t-1} + \mathbb{P}_{t-1}, \\ &\ldots \\ \gamma_{t,t-i} &= \mathbb{A}_{t-1} \gamma_{t-1,t-i} + \mathbb{P}_{t-1} \Lambda^{i-1}, \\ &\ldots \\ \gamma_{t,1} &= \mathbb{A}_{t-1} \gamma_{t-1,1} + \mathbb{P}_{t-1} \Lambda^{t-2} \end{aligned}$$

Indeed, inserting (B.9) into (B.8) and taking into account $z_t = \varepsilon_t + \Lambda\varepsilon_{t-1} + \cdots + \Lambda^{t-1}\varepsilon_1$, we obtain

$$\begin{aligned} s_{t+1} = \mathbb{A}_t(\gamma_{t,t}\varepsilon_t + \gamma_{t,t-1}\varepsilon_{t-1} + \cdots + \gamma_{t,2}\varepsilon_2 + \gamma_{t,1}\varepsilon_1) \\ +\mathbb{P}_t(\varepsilon_t + \Lambda\varepsilon_{t-1} + \cdots + \Lambda^{t-1}\varepsilon_1) + \mathbb{R}_{1,t}\varepsilon_{t+1}, \end{aligned} \tag{B.10}$$

Collecting terms with ε_j gives

$$\begin{aligned} s_{t+1} = \mathbb{R}_{1,t}\varepsilon_{t+1} + (\mathbb{A}_t\gamma_{t,t} + \mathbb{P}_t)\varepsilon_t \\ +(\mathbb{A}_t\gamma_{t,t-1} + \mathbb{P}_t\Lambda)\varepsilon_{t-1} + \cdots + (\mathbb{A}_t\gamma_{t,1} + \mathbb{P}_t\Lambda^{t-1})\varepsilon_1. \end{aligned}$$

Thus, for each t we compute $\gamma_{t,i}$, starting with the first index $t = 1$, then decreasing the index $i = t, t-1, \ldots, 1$ and using at each step $\gamma_{t-1,i}$. For the variable $u_t^{(1)}$ we also have a moving-average representation. Inserting the moving-average representation of the process $z_t^{(1)}$ and (B.9) in (B.6), we have

$$u_t^{(1)} = -K_{T,t}(\gamma_{t,t}\varepsilon_t + \cdots + \gamma_{t,1}\varepsilon_1) - R_t(\varepsilon_t + \Lambda\varepsilon_{t-1} + \cdots + \Lambda^{t-1}\varepsilon_1), \tag{B.11}$$

or in the shorter form

$$u_t^{(1)} = \delta_{t,t}\varepsilon_t + \delta_{t,t-1}\varepsilon_{t-1} + \cdots + \delta_{t,2}\varepsilon_2 + \delta_{t,1}\varepsilon_1, \tag{B.12}$$

where $\delta_{t,i} = -K_{T,t}\gamma_{t,i} - R_t\Lambda^{i-1}$.

Taking into account that $x_t^{(1)} = Z_{11}s_t^{(1)} + Z_{12}u_t^{(1)}$ and $y_t^{(1)} = Z_{21}s_t^{(1)} + Z_{22}u_t^{(1)}$, we get the moving-average representation for original variables

$$x_t^{(1)} = \rho_{t,t}^x\varepsilon_t + \rho_{t,t-1}^x\varepsilon_{t-1} + \cdots + \rho_{t,2}^x\varepsilon_2 + \rho_{t,1}^x\varepsilon_1,$$

$$y_t^{(1)} = \rho_{t,t}^y\varepsilon_t + \rho_{t,t-1}^y\varepsilon_{t-1} + \cdots + \rho_{t,2}^y\varepsilon_2 + \rho_{t,1}^y\varepsilon_1,$$

where $\rho_{t,i}^x = Z_{11}\gamma_{t,i} + Z_{12}\delta_{t,i}$ and $\rho_{t,i}^y = Z_{21}\gamma_{t,i} + Z_{22}\delta_{t,i}$.

Appendix C. Series expansion for Burnside's model

Substituting (63) and (64) into (60) yields

$$\begin{aligned} &y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)} + \cdots \\ &= \beta E_t\Big\{ \exp\left[\theta\left(x_{t+1}^{(0)} + \sigma x_{t+1}^{(1)}\right)\right]\left[1 + y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y^{(2)} + \cdots\right]\Big\} \end{aligned}$$

Expanding exponential for small σ

$$
\begin{aligned}
&y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)} + \cdots \\
&= \beta E_t \exp(\theta x_{t+1}^{(0)}) \left[1 + \sigma\theta x_{t+1}^{(1)} + \frac{1}{2}\left(\sigma\theta x_{t+1}^{(1)}\right)^2 + \cdots \right] \left[1 + y_{t+1}^{(0)} + \sigma y_{t+1}^{(1)} + \sigma^2 y_{t+1}^{(2)} + \cdots \right]
\end{aligned}
$$

Collecting the terms of like powers of σ of the last equation, we have

$$
\begin{aligned}
&y_t^{(0)} + \sigma y_t^{(1)} + \sigma^2 y_t^{(2)} + \cdots \\
&= \beta \exp\left(\theta x_{t+1}^{(0)}\right) E_t \big\{ (1 + y_{t+1}^{(0)}) + \sigma \left[\theta x_{t+1}^{(1)} (1 + y_{t+1}^{(0)}) + y_{t+1}^{(1)} \right] \\
&+ \sigma^2 \big[\frac{1}{2} (\theta x_{t+1}^{(1)})^2 (1 + y_{t+1}^{(0)}) + \theta x_{t+1}^{(1)} y_{t+1}^{(1)} + y_{t+1}^{(2)} + \cdots \big\}
\end{aligned}
$$

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